ADITYA ENGINEERING COLLEGE(A)

VECTOR INTEGRATION

By

Dr.B.Krishnaveni

H&BS Department

Aditya Engineering College(A)

Surampalem.

Review:

- INTRODUCTION
- INTEGRATION OF VECTORS

Objectives:

- LINE INTEGRAL
- WORK DONE
- PRACTICE PROBLEMS

Integration of vectors

If two vector functions $\overline{F}(t)$ and $\overline{G}(t)$ be such that $\frac{d\overline{G}(t)}{dt} = \overline{F}(t)$ then $\overline{G}(t)$ is called an indefinite integral of $\overline{F}(t)$ with respect to the scalar variable t and can be written as

$\int \overline{F}(t)dt = \overline{G}(t) + c$ where c is an arbitrary constant vector.

It's definite integral is

$$\int_a^b \overline{F}(t) = [\overline{G}(t)+c]_a^b = \overline{G}(b) - \overline{G}(a)$$

Problem:

If $\overline{F}(t) = (5t^2 - 3t)\bar{i} + 6t^3\bar{j} - 7t\bar{k}$ then evaluate $\int_2^4 \overline{F}(t)dt$

Sol:

$$\int_2^4 \overline{F}(t)dt = \int_2^4 [(5t^2 - 3t)\bar{i} + 6t^3\bar{j} - 7t\bar{k}]dt$$

$$= \left[\left(\frac{5t^3}{3} - \frac{3t^2}{2}\right)\bar{i} + \frac{3t^4}{2}\bar{j} - \frac{7t^2}{2}\bar{k}\right]_2^4$$

$$= \left(\frac{5(64-8)}{3} - \frac{3(16-4)}{2}\right)\bar{i} + \frac{3(256-16)}{2}\bar{j}$$

$$- \frac{7(16-4)}{2}\bar{k}$$

$$= \frac{226}{3}\bar{i} + 360\bar{j} - 42\bar{k}$$

Line Integral

If $\overline{F}(r) = f\overline{i} + g\overline{j} + h\overline{k}$ where f, g,h are functions of x, y, z and $d\overline{r} = dx\overline{i} + dy\overline{j} + dz\overline{k}$,then

$$\int_c F \cdot d\overline{r} = \int_c (fdx + gdy + hdz)$$

is called line integral of $\overline{F}$ over c, where c is an curve in space

Workdone by a Force

If $\overline{F}$ represents the force acting on a particle moving along an arc AB, then the total workdone by force $\overline{F}$ during the displacement from A to B given by line integral

$$\int_A^B \overline{F} \cdot d\overline{r}$$

i.e., if $\overline{F} = f\overline{i} + g\overline{j} + h\overline{k}$ and $\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$

$$\Rightarrow d\overline{r} = dx\overline{i} + dy\overline{j} + dz\overline{k}$$

$$\therefore \int_A^B \overline{F} \cdot d\overline{r} = \int_A^B fdx + gdy + hdz$$

Problem:

Find the workdone by a force $\overline{F} = (x^2 - y^2 + x)\bar{i} - (2xy + y)\bar{j}$ which moves a plane in xy-plane from (0,0) to (1,1) along the parabola $y^2 = x$

Sol: Given parabola $y^2 = x$ in xy-plane

Hence z=0

Workdone= $\int_c \overline{F} \cdot d\bar{r} = \int_c (fdx + gdy)$

$$= \int_c (x^2 - y^2 + x)dx + -(2xy + y)dy$$

$$= \int_0^1 (y^4 - y^2 + y^2)2ydy - (2y^3 + y)dy$$

$$= \int_0^1 (2y^5 - 2y^3 - y)dy$$

$$= [2\frac{y^6}{6} - 2\frac{y^4}{4} - \frac{y^2}{2}]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} - \frac{1}{2}$$

$$= -\frac{2}{3}$$

Problem 2:

Find the workdone by a force $\overline{F} = z\overline{i} + x\overline{j} + y\overline{k}$,when it moves a particle along the arc of the curve $\overline{r} = \cos t\overline{i} + \sin t\overline{j} - t\overline{k}$ from

$t = 0$ to $t = 2\pi$

Sol: Given $\overline{r} = \cos t\overline{i} + \sin t\overline{j} - t\overline{k}$

Hence $x = \cos t, y = \sin t, z = -t$

now,

$$d\overline{r} = (-\sin \overline{i} + \cos t\overline{j} - \overline{k})dt$$

$$\overline{F} \cdot d\overline{r} = (-t\overline{i} + \cos t\overline{j} + \sin t\overline{k}) \cdot (-\sin t\overline{i} + \cos t\overline{j} - \overline{k})dt$$

$$= (t\sin t + \cos^2 t - \sin t)dt$$

Workdone=

$$\int_c \overline{F}\cdot d\overline{r} = \int_0^{2\pi}(t\sin t+\cos^2 t-\sin t)dt$$

$$=\int_0^{2\pi} t\sin t dt+\int_0^{2\pi}\left(\frac{1+\cos 2t}{2}\right)dt-\int_0^{2\pi}\sin t dt$$

$$=[t(-\cos t)-(-\sin t)]_0^{2\pi}+\left[\frac{1}{2}t+\frac{\sin 2t}{4}\right]_0^{2\pi}-(-\cos t)_0^{2\pi}$$

$$=-2\pi+\frac{1}{2}(2\pi)+(1-1)$$

$$=-2\pi+\pi \quad =-\pi$$

Problem 3:

Find the workdone by a force $\overline{F} = 3x^2\overline{i} + (2xz - y)\overline{j} + z\overline{k}$ along the straight line from (0,0,0) to (2,1,3)

Sol: Given, $\overline{F} = 3x^2\overline{i} + (2xz - y)\overline{j} + z\overline{k}$

Equation of line OA is

$$\frac{x-0}{2} = \frac{y-0}{1} = \frac{z-0}{3} = t(say)$$

$$\Rightarrow x = 2t, y = t, z = 3t$$

$$\therefore t = 0 \ \ to \ \ t = 1$$

Workdone along the line O(0,0,0) to A(2,1,3) is given by

$$\int_{0A} \bar{F}\cdot d\bar{r} = \int_{OA} 3x^2dx + (2xz - y)dy + zdz$$

$$= \int_{t=0}^{1} 3(4t^2)2dt + (12t^3 - t)dt + 3t.3dt$$

$$= \int_{t=0}^{1} (36t^2 + 8t)dt$$

$$= \left[\frac{36t^3}{3} + \frac{8t^2}{2}\right]_0^1$$

$$= 12 + 4 = 16$$

Problem 4:

Find the workdone by a force $\overline{F} = xy\overline{i} + yz\overline{j} + zx\overline{k}$ along the curve $x = 1, y = t, z = t^2$ from t=0 to t=1

Sol: Given, $\overline{F} = xy\overline{i} + yz\overline{j} + zx\overline{k}$

and curve $x = 1, y = t, z = t^2 \Rightarrow dx = 0, dy = 1, dz = 2tdt$

$$\overline{F} \cdot d\overline{r} = (xy\overline{i} + yz\overline{j} + zx\overline{k}) \cdot (dx\overline{i} + dy\overline{j} + dz\overline{k})$$

$$= xydx + yzdy + zxdz$$

$$= t(0) + t^3dt + t^2(2tdt) = 3t^3dt$$

- Work done=

$$\int_c \overline{F} \cdot d\overline{r} = \int_{t=0}^{1} 3t^3 dt$$

$$= \left[\frac{3t^4}{4}\right]_0^1$$

$$= \frac{3}{4} - 0$$

$$= \frac{3}{4}$$

Conservative force field

If $\int \overline{F} \cdot d\overline{r} = 0$,the field is cosevative, i.e., no work done in displacement from a point **a** to another point and back to **a**(i.e., work done is independent of the path .

Hence every irrotational vector is consevative and there exists a scalar ϕ such that $\overline{F} = \nabla \phi$ and this ϕ is called scalar potential

PROBLEM:

Show that $\bar{F} = (2xy + z^3)\bar{i} + x^2\bar{j} + 3xz^2\bar{k}$ is conservative and find work done by a moving particle from (0,0,0) to (1,1,1)

Solution: Given,

$$\bar{F} = (2xy + z^3)\bar{i} + x^2\bar{j} + 3xz^2\bar{k}$$

$$curl F = \nabla \times F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$curl\ F = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \bar{i}\left[\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2)\right] - \bar{j}\left[\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3)\right] + \bar{k}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3)\right]$$

$$= \bar{i}\ [0 - 0] - \bar{j}\,[3z^2 - 3z^2] + \bar{k}[2X - 2X]$$

$$= \bar{0}.$$

Curl F= $\bar{0}$.

So, F is irrotational, hence F is conservative

A vector F is conservative if their exists a scalar function $\emptyset$ such that F=$\nabla\emptyset$.

Let $\emptyset(x,y,z)$ be a scalar function then,

$$\nabla\emptyset = \frac{\partial\emptyset}{\partial x}\bar{i} + \frac{\partial\emptyset}{\partial y}\bar{j} + \frac{\partial\emptyset}{\partial z}\bar{k}$$

$$F = \frac{\partial\emptyset}{\partial x}\bar{i} + \frac{\partial\emptyset}{\partial y}\bar{j} + \frac{\partial\emptyset}{\partial z}\bar{k}$$

$$(2xy+z^3)\bar{i}+x^2\bar{j}+3xz^2\bar{k}=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

Comparing on both sides ,we get

$$\frac{\partial\phi}{\partial x}=(2xy+z^3)\Rightarrow\phi=x^2y+xz^3+f_1(y,z)$$

$$\frac{\partial\phi}{\partial y}=x^2\Rightarrow\phi=x^2y+f_2(x,z)$$

$$\frac{\partial\phi}{\partial z}=3xz^2\Rightarrow\phi=xz^3+f_3(x,y)$$

$$\therefore\phi=x^2y+xz^3+c$$

which is our required scalar potential

Workdone from O(0,0,0) to A(1,1,1) is given by

$$\int_{0A} \overline{F} \cdot d\overline{r} = \int_{OA} (2xy + z^3)dx + x^2 dy + 3xz^2 dz$$

$$= \int_{O}^{A} d(x^2 y + xz^3)$$

$$= [x^2 y + xz^3]_{(0,0,0)}^{(1,1,1)}$$

$$= 1 + 1 = 2units$$

Practice Problem

1.Find the workdone by a force $\overline{F} = 3x^2y\bar{i} + 2y\bar{j} + 4xz^2\overline{k}$ along the curve $x = t, y = t^2, z = t^3$ from t=0 to t=1

2. Show that $F = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\overline{k}$ is conservative and find work done by a moving particle from (1,-1,2) to (3,2,-1)

Review:

- INTRODUCTION
- INTEGRATION OF VECTORS
- WORK DONE

Objectives:

- LINE INTEGRAL
- PRACTICE PROBLEMS

CIRCULATION:-

If $\bar{v}$ represents the velocity of a fluid particle and C is a closed curve, then the integral $\oint_c \bar{v} \,.\, d\bar{r}$ is called the circulation of $\bar{v}$ round the curve C.

If $\int_c \bar{v}.d\bar{r} = 0$, then the field $\bar{v}$ is called conservative, i.e., no work is done and the energy is conserved..

1) If $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\,\bar{j}$ evaluate $\oint_c \bar{F}\,.\,\overline{dr}$ where curve c is the rectangle in xy plane bounded by y=0,y=b,x=0,x=a.

Sol:)

Given $\bar{F} = (x^2 + y^2)\bar{i} - 2xy\,\bar{j}$

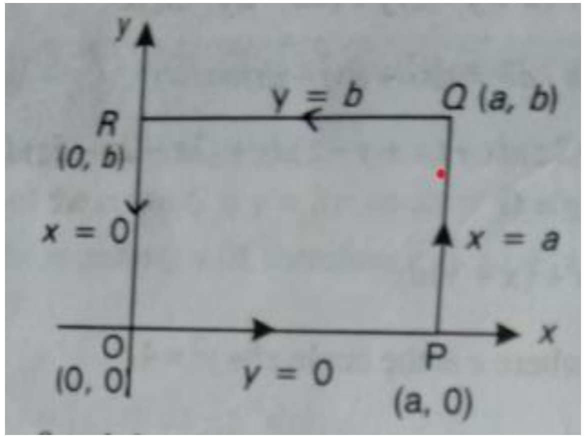
$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F}.d\bar{r} = (x^2 + y^2)dx - 2xy\,dy$$

$$\int_c \bar{F}.d\bar{r} = \int_c (x^2 + y^2)dx - 2xy\,dy$$

$$\int_{c} F.dr = \int_{0P} F.dr + \int_{PQ} F.dr + \int_{QR} F.dr + \int_{R0} F.dr \quad - \quad (1)$$

(i)Along the line OP:



y=0 and dy=0 and x varies from 0 to a.

$$\therefore \int_{OP} \bar{F}.d\bar{r} = \int_0^a x^2 \ dx = \frac{a^3}{3}$$

(ii)Along PQ:

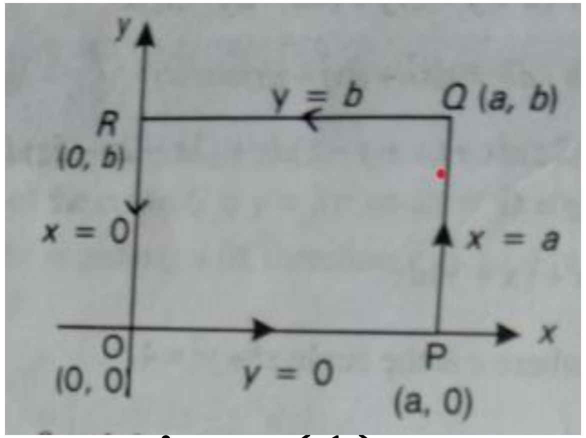
x=a => dx=0 and y changes from 0 to b.

$$\therefore \int_{PQ} \bar{F}.d\bar{r} = \int_0^b (-2ay)dy = -a\, b^2$$

(iii) Along QR:

We have y=b => dy=0 and x changes from a to 0.

$$\int_{QR} \bar{F}\,.\,d\bar{r} = \int_a^0 (x^2 + b^2)dx = \frac{-a^3}{3} - ab^2$$



(iv) Along R0:

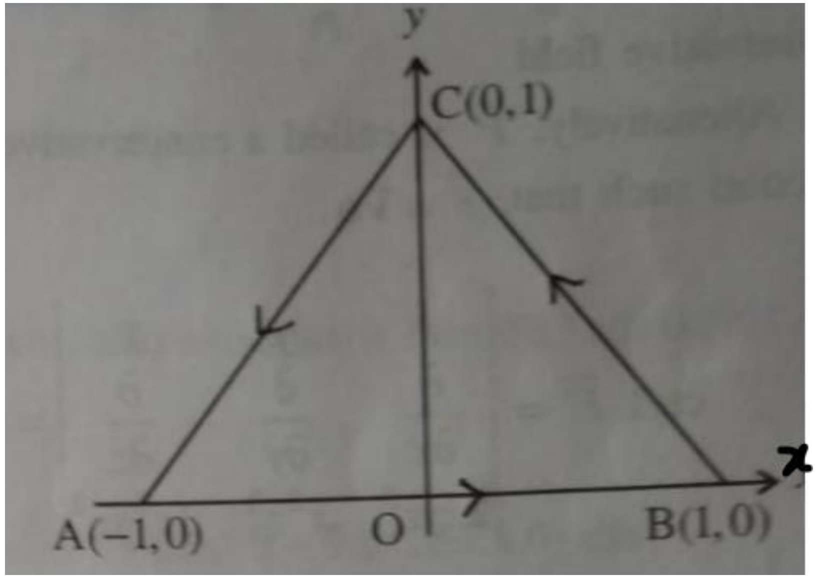
x=0 =>dx=0 and y varies from b to 0.

$$\therefore \int_{R0} \bar{F}.\,d\bar{r} = \int_b^0 0\,dy = 0$$

Hence substituting (i),(ii),(iii) and (iv) in equation (1) we get,

$$\oint_c \bar{F}.\,d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{1}{3}a^3 - ab^2 = -2ab^2 \ldots$$

2) Compute the line integral $\int (y^2 dx - x^2 dy)$ round the triangle whose vertices are (1,0),(0,1),(-1,0) in the xy-plane.



Sol:) Let A=(-1,0), B=(1,0),C=(0,1)

Equation of AB (x-axis) is y=0

Equation of BC is x+y=1

Equation of CA is y-x=1

$\therefore \int_C (y^2 dx - x^2 dy) =$

$$\int_{AB} + \int_{BC} + \int_{CA} - (1)$$

(i)Along the line AB:

y=0 => dy=0

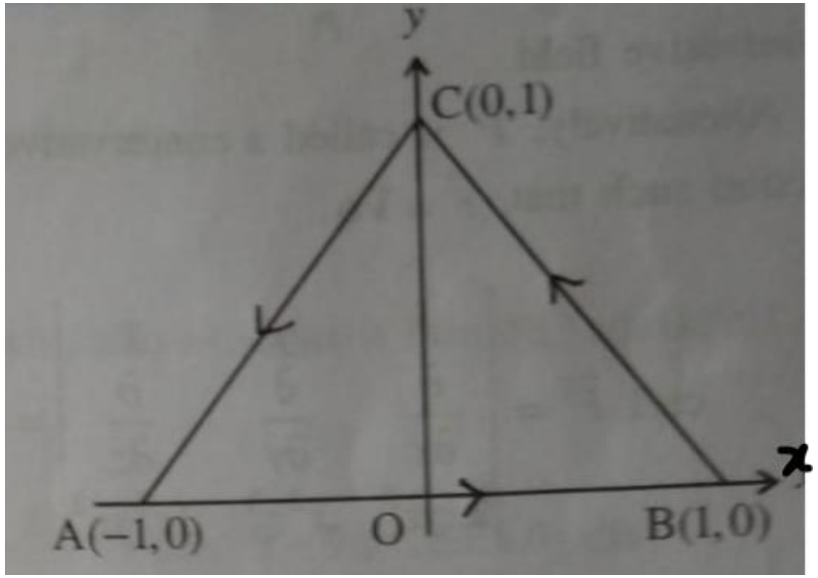
$$\therefore \int_{AB} = \int 0 = 0$$

(ii) Along the line BC:

x + y =1 => y =1 – x   ∴ dy= - dx

$$\therefore \int_{BC} = \int_{x=1}^{0} (1-x)^2\, dx - x^2(-dx)$$

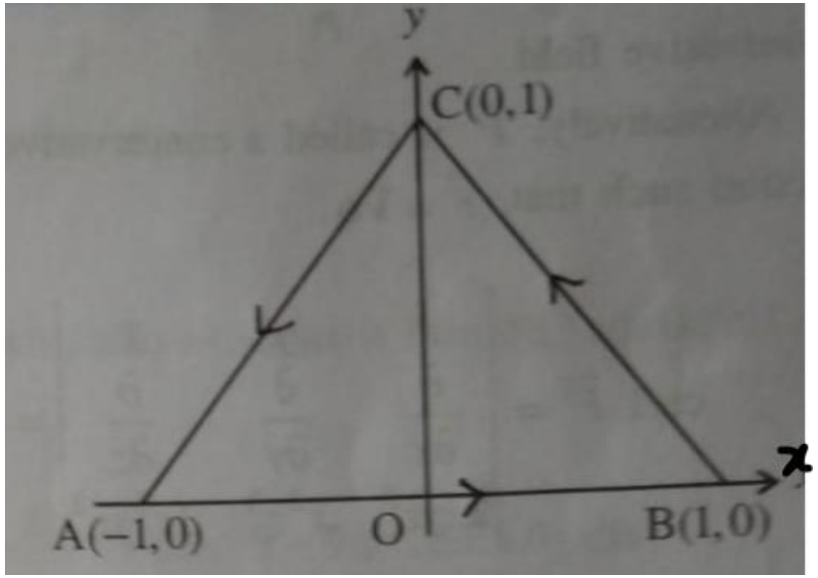
$$= \left[\frac{-1(1-x)^3}{3}\right]_{x=1}^{0} + \left[\frac{x^3}{3}\right]_{x=1}^{0} = \frac{-1}{3} - \frac{1}{3} = \frac{-2}{3}$$



(iii)Along the line CA:

$$\int_{CA} = \int_{x=0}^{-1} (1+x)^2 dx - x^2 dx = \left[\frac{(1+x)^3}{3} - \frac{x^3}{3}\right]_0^{-1}$$

$=0+\frac{1}{3}+\frac{1}{3}+0=0$



Hence the required line integral

= 0 - 2/3 + 0

= -2/3 [using(1)]…

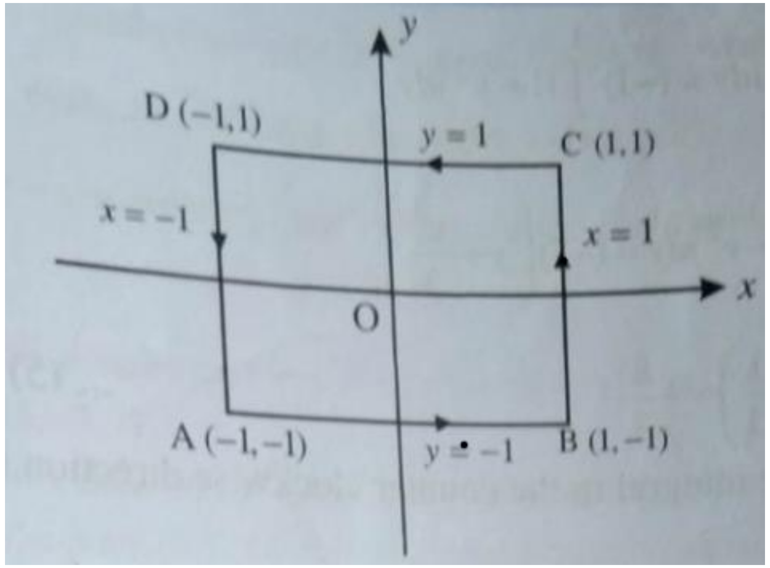
3) Evaluate the line integral $\int_c [(x^2 + xy)dx + (x^2 + y^2)dy]$

where c is the square formed by the lines x=$\pm$1 and y =$\pm$1.

Sol:)

Here $\int_c \bar{F}.d\bar{r} = \int_c [(x^2 + xy)dx + (x^2 + y^2)dy]$

In the counter clock-wise direction



$$\int_c \bar{F}.d\bar{r} = \int_{AB} \bar{F}.d\bar{r} + \int_{BC} \bar{F}.d\bar{r} + \int_{CD} \bar{F}.d\bar{r} + \int_{DA} \bar{F}.\bar{d}r \ \ldots(1)$$

Along AB:

Here y= -1. ∴dy=0.

$$\therefore \int_{AB} \bar{F}.d\bar{r} = \int_{-1}^{1}(x^2 - x)dx = \int_{-1}^{1} x^2dx - \int_{-1}^{1} x\,dx$$

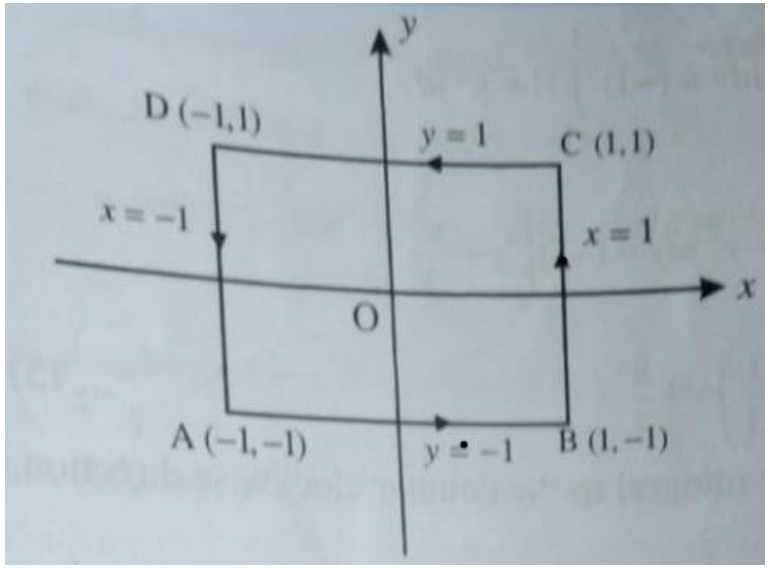
$$=2\int_0^1 x^2dx - 0 = 2\left[\frac{x^3}{3}\right]_1^0 = \frac{2}{3} \quad \ldots(2)$$

Along BC:

Here x =1. $\therefore dx = 0.$

$\therefore \int_{BC} \bar{F}.d\bar{r} = \int_{-1}^{1}(1+y)^2\,\mathrm{dy}$

$=2\left[y+\frac{y^3}{3}\right]_0^1=2[1+\frac{1}{3}]=\frac{8}{3}$ ... (3)



Along CD:

Here y=1. ∴ dy=0.

$\therefore \int_{CD} \bar{F}.d\bar{r} = \int_{1}^{-1}(x^2+x)dx$

$= (-1)\int_{-1}^{1}(x^2+x)dx =$

$=(-1)\left[2\int_0^1 x^2\,dx+0\right] = -\frac{2}{3}$ .....(4)

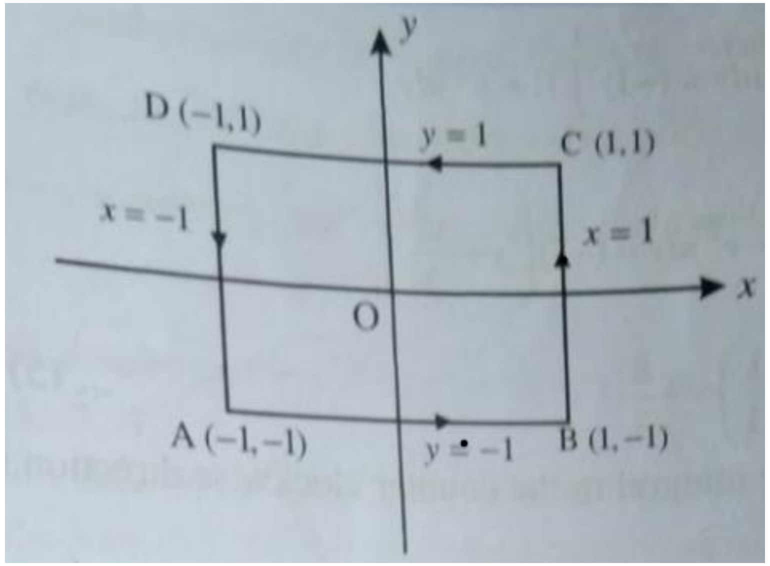
Along DA:

Here x= -1. ∴ dx=0.

$$\therefore \int_{DA} \bar{F}.d\bar{r} = \int_{1}^{-1} (1+y^2)dy$$

$$=(-1)\int_{-1}^{1}(1+y^2)dy = (-2)\int_{0}^{1}(1+y^2)dy$$

$$=(-2)\left[y+\frac{y^3}{3}\right]_0^1 = (-2)[1+\frac{1}{3}] = -\frac{8}{3} \quad \dots (5)$$



Hence the required line integral in the counter clock-wise direction is

$$\int_c \bar{F}.d\bar{r} = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0, \text{ using (1)}.....$$

4)If $\bar{F} = 3xy\bar{i} - y^2\,\bar{j}$ evaluate $\int_c \bar{F}.d\bar{r}$ where C is the curve $y=2x^2$ in the xy plane from (0,0) to (1,2)

Sol:) Given $\bar{F} = 3xy\bar{i} - y^2\,\bar{j}$

$$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} + dz\,\bar{k}$$

$$\bar{F}.d\bar{r} = 3xy\,dx - y^2dy$$

C is the curve $y = 2x^2$

$$dy = 4x\,dx$$

$$x \;:\; 0 \rightarrow 1$$

$$\therefore \int_c \bar{F}.d\bar{r} = \int_c (3xy\,dx - y^2)dy$$

$$=\int_{x=0}^{1} 3x(2x^2)dx - 4x^4(4x)dx$$

$$=\int_0^1 (6x^3 - 16x^5)\mathrm{dx}$$

$$=\left[\frac{6x^4}{4} - \frac{16x^6}{6}\right]_0^1$$

$$=\frac{6}{4} - \frac{16}{6} = -\frac{7}{6}\ldots\ldots.$$

5)If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\,\bar{j}$, evaluate $\int_c \bar{F}.d\bar{r}$ along the curve C in the xy plane $y=x^3$ from (1,1) to (2,8)

Sol:) Given $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\,\bar{j}$ ....(1)

Along the curve $y=x^3$, $dy=3x^2\,dx$

$\therefore \bar{F} = (5x^4 - 6x^2)\bar{i} + (2x^3 - 4x)\,\bar{j}$ [Putting $y=x^3$ in (1)]

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j} = dx\,\bar{i} + 3x^2dx.\bar{j}$

$\therefore \bar{F}.d\bar{r} = [(5x^4 - 6x^2)\bar{i} + (2x^3 - 4x)\,\bar{j}].(dx\,\bar{i} + 3x^2dx.\bar{j})$

$= (5x^4 - 6x^2)dx + (2x^3 - 4x)\,3x^2dx$

$= (6x^5+5x^4-12x^3-6x^2)dx$

Hence $\int_c \bar{F}.d\bar{r} = \int_1^2 (6x^5+5x^4-12x^3-6x^2)dx$

$= \left[6\frac{x^6}{6} + 5\frac{x^5}{5} - 12\frac{x^4}{4} - 6\frac{x^3}{3}\right]$

$= [x^6 + x^5 - 3x^4 - 2x^3]_1^2$

$=16(4+2-3-1) - (1+1-3-2)$

$= 32+3=35….$

Surface integrals :

Let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ where F_1, F_2, F_3 are continuous and differentiable functions of x, y, z.

Then surface integral is $\int \bar{F}.\bar{n}\, dS$

Where $\bar{n}$ is the unit outward normal vector

Along xy-plane normal vector is $\bar{k}$

Along yz-plane normal vector is $\bar{i}$

Along zx-plane normal vector is $\bar{j}$

NOTE:

Let R_1 be the projection of S on xy- plane . Then

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_1} \frac{\bar{F}.\bar{n}\, dxdy}{|\bar{n}.\bar{k}|}$$

Similarly,

Let R_2 be the projection of S on yz- plane . Then

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\bar{F}.\bar{n}\, dydz}{|\bar{n}.\bar{\imath}|}$$

Let R_3 be the projection of S on zx- plane . Then

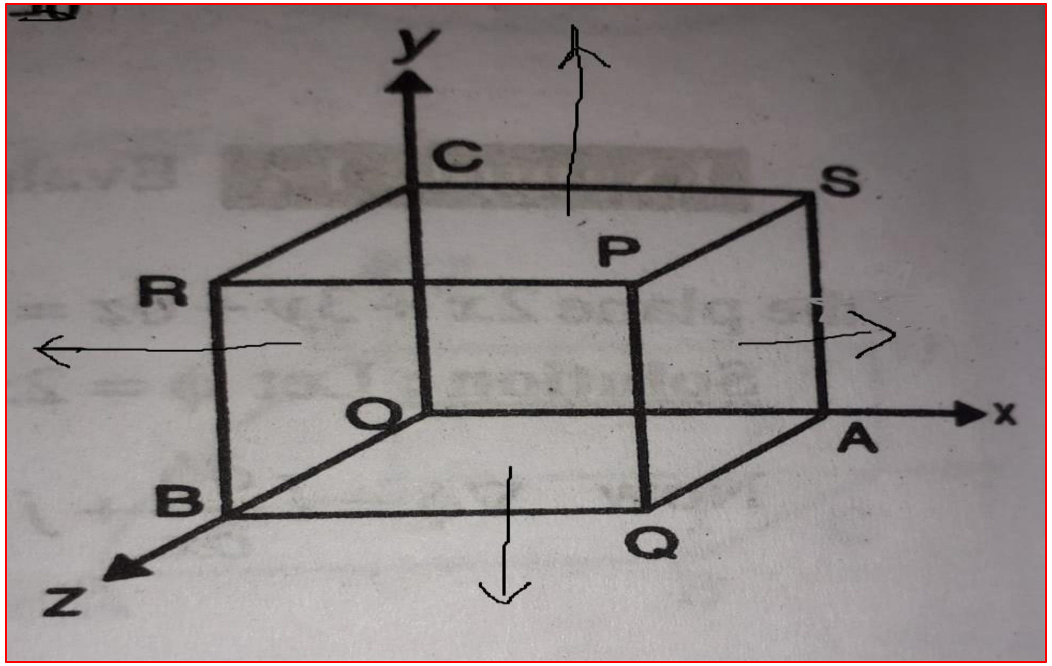
$$\int_S \bar{F}.\bar{n}dS = \iint_{R_3} \frac{\bar{F}.\bar{n}\, dzdx}{|\bar{n}.\bar{\jmath}|}$$

1) If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$, evaluate $\int \bar{F}.\bar{n}\, dS$ where S is the surface of the cube bounded by x=0, x=a, y=0, y=a, z=0, z=a.

Sol: consider the volume within the cube PQASCRBO in figure bounded by x=0, x=a, y=0, y=a, z=0, z=a.

Here $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

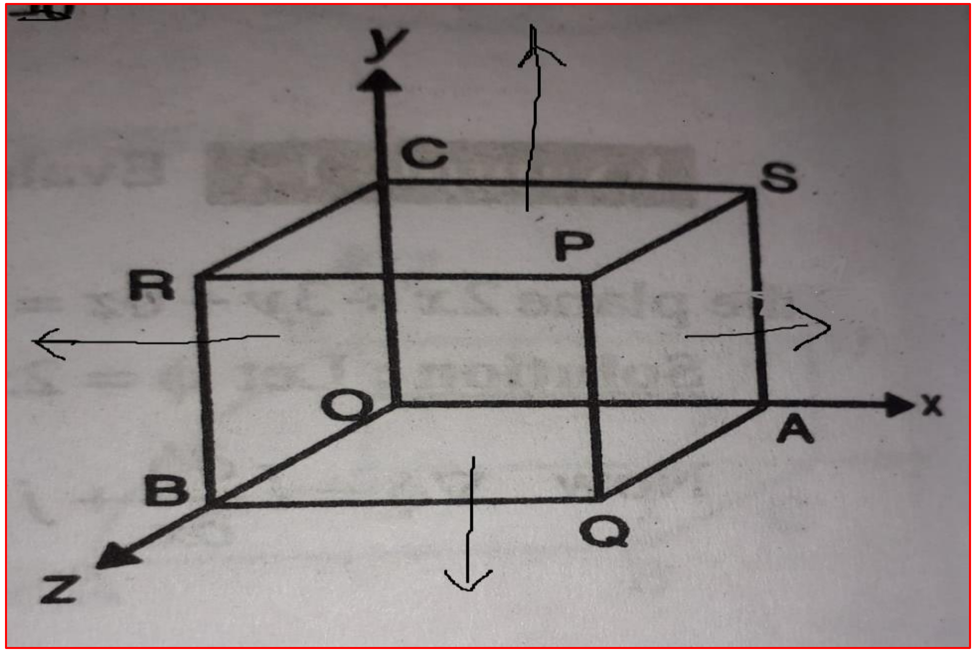
Let us calculate $\int_S \bar{F}.\bar{n}\, dS$ for each face of the cube.



I) Along the face R_1 =OCRB, it is in yz-plane

X=0, ds= dydz, $\bar{n} = -\bar{i}$

$0 \le y \le a, 0 \le z \le a$



$\bar{F}.\bar{n} = -4xz = 0 \ (since\ x = 0)$

$$\iint_{R_1} \bar{F}.\bar{n}\, dS = 0$$

II)Along the face $R_2 = \mathrm{ASPQ}$, it is in yz-plane

X=a, ds= dydz, $\bar{n} = \bar{i}$

$0 \le y \le a, 0 \le z \le a$

$\bar{F}.\bar{n} = 4xz = 4az \quad (since\ x = a)$

$$\iint_{R_2} \bar{F}.\bar{n}\, dS = \int_{y=0}^{a}\int_{z=0}^{a} 4az\, dydz = 4a\int_{y=0}^{a}\left[\frac{z^2}{2}\right]_0^a dy = 4a.\frac{a^2}{2}\ .\ [y]_0^a = 2a^4$$

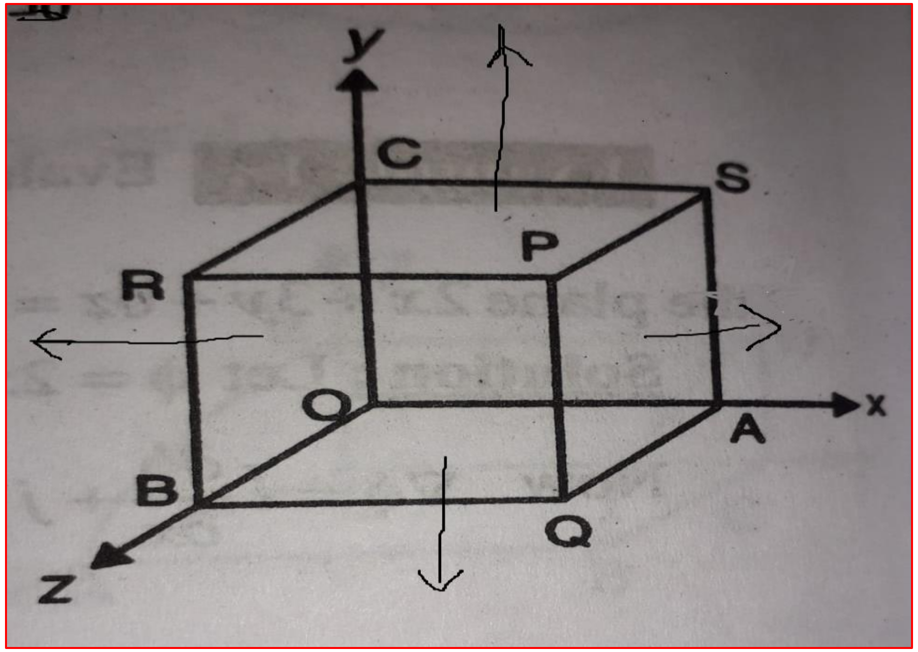
III) Along the face R_3 =OAQB, it is in xz-plane

y=0, ds= dxdz, $\bar{n} = -\bar{j}$

$0 \le x \le a, 0 \le z \le a$

$\bar{F}.\bar{n} = y^2 = 0\ (since\ y = 0)$

$$\iint_{R_3} \bar{F}.\bar{n}\, dS = 0$$



IV)Along the face $R_4 = \text{CSPR}$, it is in xz-plane

y=a, ds= dxdz, $\bar{n} = \bar{j}$

$0 \le x \le a, 0 \le z \le a$

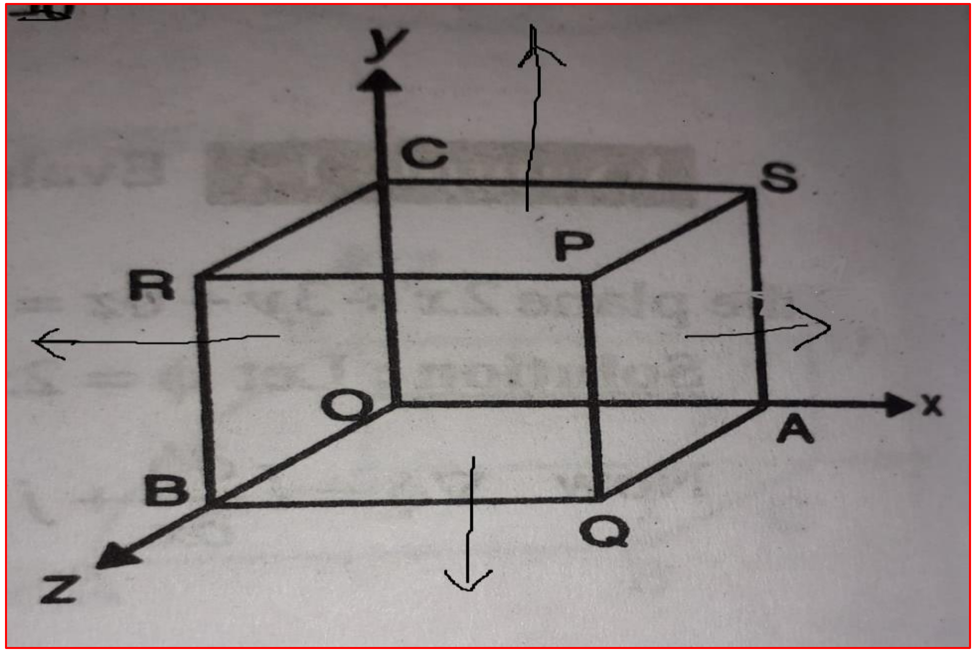
$\bar{F}.\bar{n} = -y^2 = -a^2 (since\ y = a)$

$$\iint_{R_4} \bar{F}.\bar{n}\, dS = \int_{x=0}^{a} \int_{z=0}^{a} (-a^2)dxdz = -a^2 \int_{x=0}^{a} [z]_0^a dx = -a^2.a\ \ [x]_0^a = -a^4$$

V) Along the face R_5 =OASC, it is in xy-plane

z=0, ds= dxdy, $\bar{n} = -\bar{k}$

$0 \leq x \leq a, 0 \leq y \leq a$



$$\bar{F}.\bar{n} = -yz = 0\ (since\ z = 0)$$

$$\iint_{R_5} \bar{F}.\bar{n}\ dS = 0$$

VI)Along the face $R_6 = \mathrm{PQBR}$, it is in xy-plane

z=a, ds= dxdy, $\bar{n} = \bar{k}$

$0 \leq x \leq a, 0 \leq y \leq a$

$$\bar{F}.\bar{n} = yz = ay \quad (since\ z = a)$$

$$\iint_{R_6} \bar{F}.\bar{n}\ dS = \int_{x=0}^{a}\int_{y=0}^{a} ay\ dxdy = a\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^a dx = a.\frac{a^2}{2}\ .\ [x]_0^a = \frac{a^4}{2}$$

$$\int \bar{F}.\bar{n}\, dS = \iint_{R_1} \bar{F}.\bar{n}\, dS + \iint_{R_2} \bar{F}.\bar{n}\, dS + \iint_{R_3} \bar{F}.\bar{n}\, dS + \iint_{R_4} \bar{F}.\bar{n}\, dS + \iint_{R_5} \bar{F}.\bar{n}\, dS + \iint_{R_6} \bar{F}.\bar{n}\, dS$$

$$\int \bar{F}.\bar{n}\, dS = 0 + 2a^4 + 0 - a^4 + 0 + \frac{a^4}{2}$$

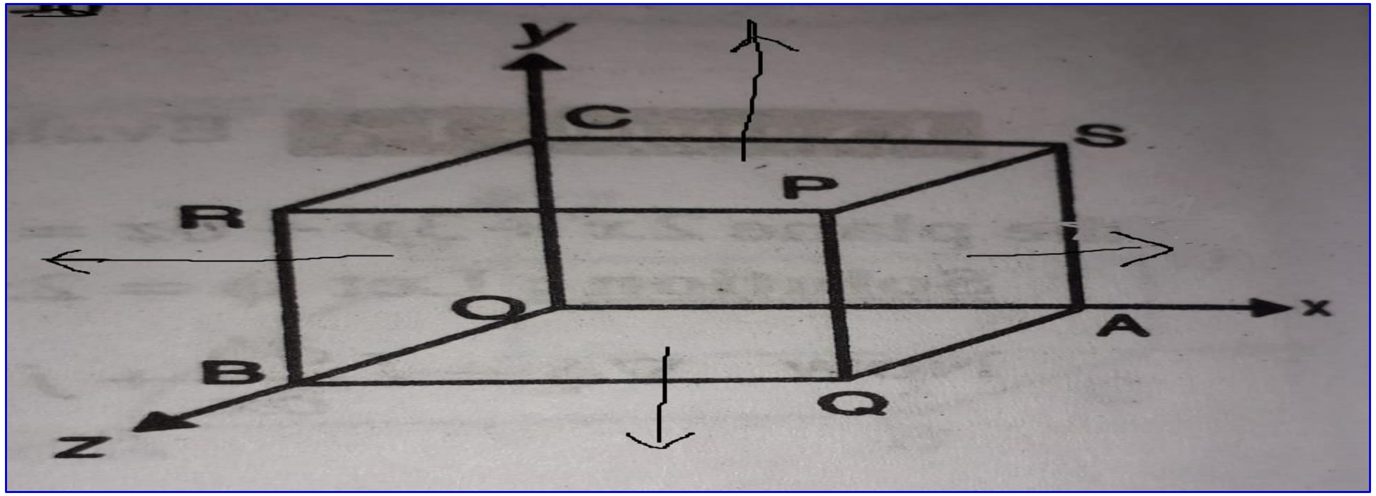
$$= \frac{3a^4}{2}$$

2) If $\bar{F} = 2xy\bar{\imath} + yz^2\bar{\jmath} + xz\bar{k}$, evaluate $\int \bar{F}.\bar{n}\, dS$ where S is the surface of the parallelepiped bounded by x=0, x=2, y=0, y=1, z=0, z=3.

Sol: consider the volume within the cube PQASCRBO in figure bounded by x=0, x=2, y=0, y=1, z=0, z=3.

Here $\bar{F} = 2xy\bar{\imath} + yz^2\bar{\jmath} + xz\bar{k}$

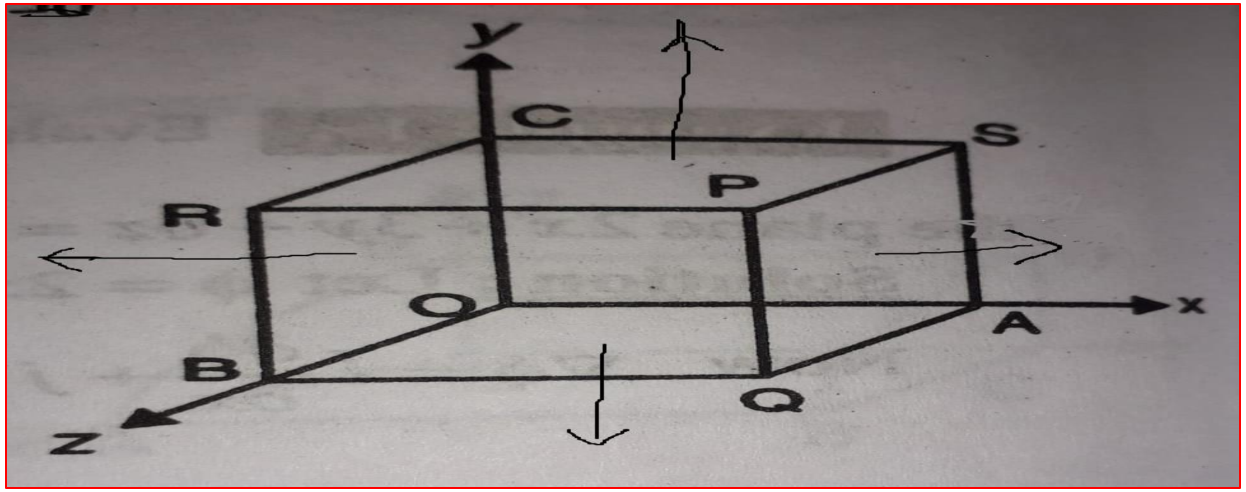
Let us calculate $\int_S \bar{F}.\bar{n}\, dS$ for each face of the cube.



I) Along the face R_1 =OCRB, it is in yz-plane

X=0, ds= dydz, $\bar{n} = -\bar{i}$

$0 \leq y \leq 1, 0 \leq z \leq 3$



$\bar{F}.\bar{n} = 2xy = 0 \ (since\ x = 0)$

$$\iint_{R_1} \bar{F}.\bar{n}\ dS = 0$$

II)Along the face $R_2 = ASPQ$, it is in yz-plane

X=2, ds= dydz, $\bar{n} = \bar{i}$

$0 \leq y \leq 1, 0 \leq z \leq 3$

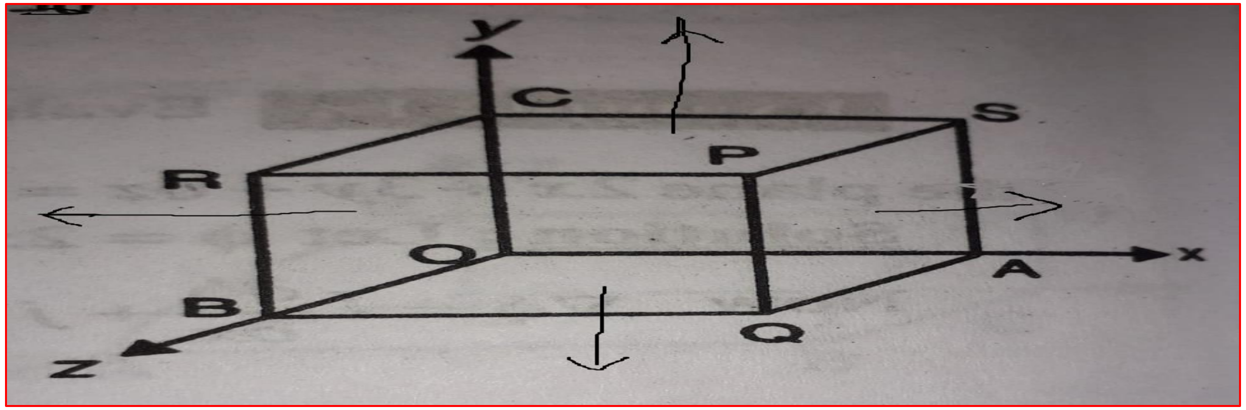
$\bar{F}.\bar{n} = 2xy = 4y \quad (since\ x = 2)$

$$\iint_{R_2} \bar{F}.\bar{n}\ dS = \int_{y=0}^{1}\int_{z=0}^{3} 4y\ dydz = 4\int_{z=0}^{a}\left[\frac{y^2}{2}\right]_0^1 dz = 4.\frac{1}{2}\ .\ [z]_0^3 = 2.3 = 6$$

III) Along the face R_3 =OAQB, it is in xz-plane

y=0, ds= dxdz, $\bar{n} = -\bar{j}$

$0 \leq x \leq 2, 0 \leq z \leq 3$



$\bar{F}.\bar{n} = -yz^2 \ (since\ y = 0)$

$$\iint_{R_3} \bar{F}.\bar{n}\, dS = 0$$

IV)Along the face $R_4 = \text{CSPR}$, it is in xz-plane

y=1, ds= dxdz, $\bar{n} = \bar{j}$

$0 \leq x \leq 2, 0 \leq z \leq 3$

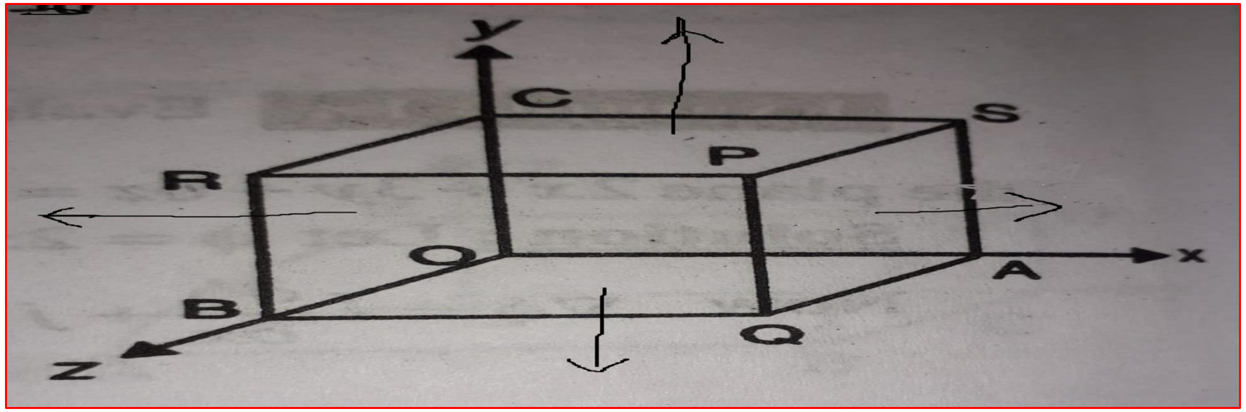
$\bar{F}.\bar{n} = yz^2 = z^2 (since\ y = 1)$

$$\iint_{R_4} \bar{F}.\bar{n}\, dS = \int_{x=0}^{2}\int_{z=0}^{3}(z^2)dxdz = \int_{x=0}^{2}\left[\frac{z^3}{3}\right]_0^3 dx = 9 \quad [x]_0^2 = 9(2) = 18$$

V) Along the face R_5 =OASC, it is in xy-plane

z=0, ds= dxdy, $\bar{n} = -\bar{k}$

$0 \le x \le 2, 0 \le y \le 1$



$\bar{F}.\bar{n} = -xz = 0\ (since\ z = 0)$

$$\iint_{R_5} \bar{F}.\bar{n}\, dS = 0$$

VI)Along the face R_6 = PQBR, it is in xy-plane

z=3, ds= dxdy, $\bar{n} = \bar{k}$

$0 \le x \le 2, 0 \le y \le 1$

$\bar{F}.\bar{n} = xz = 3x \quad (since\ z = 3)$

$$\iint_{R_6} \bar{F}.\bar{n}\, dS = \int_{x=0}^{2}\int_{y=0}^{1} 3x\, dxdy = 3\int_{y=0}^{1}\left[\frac{x^2}{2}\right]_0^2 dx = 3.\frac{2^2}{2}\ .\ [y]_0^1 = 6$$

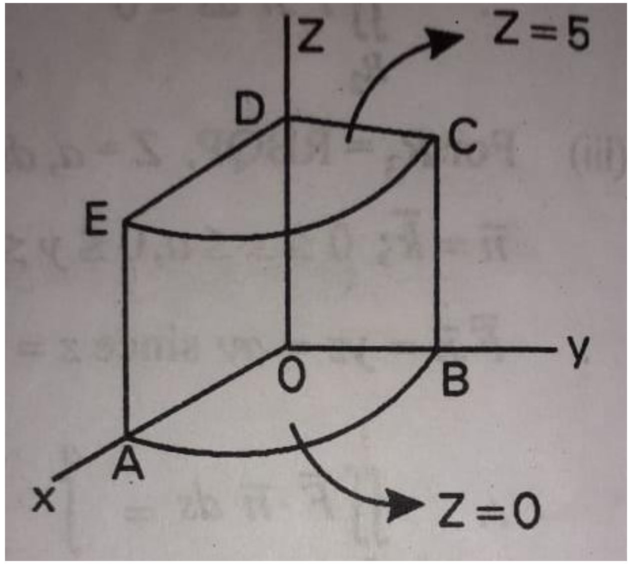
$$\int \bar{F}.\bar{n}\, dS = \iint_{R_1} \bar{F}.\bar{n}\, dS + \iint_{R_2} \bar{F}.\bar{n}\, dS + \iint_{R_3} \bar{F}.\bar{n}\, dS + \iint_{R_4} \bar{F}.\bar{n}\, dS + \iint_{R_5} \bar{F}.\bar{n}\, dS + \iint_{R_6} \bar{F}.\bar{n}\, dS$$

$$\int \bar{F}.\bar{n}\, dS = 0 + 6 + 0 + 18 + 0 + 6$$

$$= 30$$

3) Evaluate $\int \bar{F}.\bar{n}\, dS$, where $\bar{F} = z\bar{\imath} + x\bar{\jmath} - 3y^2 z\bar{k}$ and S is the curved surface $x^2 + y^2 = 16$ included in the first octant between z=0 and z=5.

Sol: Given surface S is the curved surface ABCEA.



Let $\emptyset = x^2 + y^2 - 16$

Normal to the surface S is grad $\emptyset$

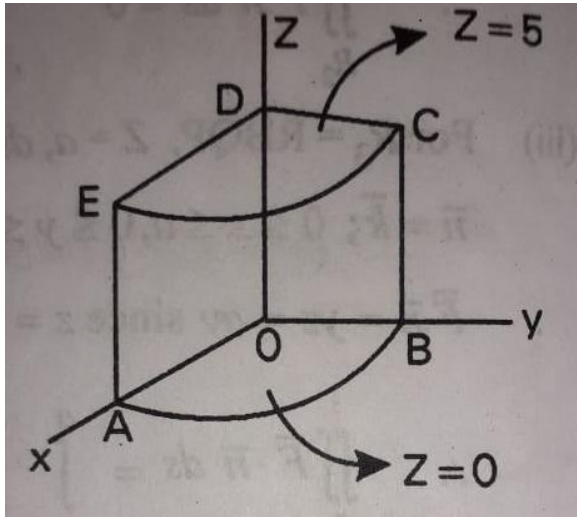
Normal vector = grad $\emptyset$= $\nabla\emptyset$ = $\bar{\imath}\frac{\partial\emptyset}{\partial x}+\bar{\jmath}\frac{\partial\emptyset}{\partial y}+\bar{k}\frac{\partial\emptyset}{\partial z}$

$= 2x\,\bar{\imath} + 2y\,\bar{\jmath} + o\,\bar{k}$

Unit normal vector is $\bar{n} = \frac{2x\,\bar{\imath}+2y\,\bar{\jmath}+o\,\bar{k}}{|2x\,\bar{\imath}+2y\,\bar{\jmath}+o\,\bar{k}|}=\frac{2x\,\bar{\imath}+2y\,\bar{\jmath}+o\,\bar{k}}{\sqrt{4x^2+4y^2+0^2}}$

$=\frac{2(x\,\bar{\imath}+y\,\bar{\jmath})}{2\sqrt{16}} = \frac{x\,\bar{\imath}+y\,\bar{\jmath}}{4}$

Consider the curved region R: OBCD



Let R be the projection of S on yz- plane . Then

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\bar{F}.\bar{n}\, dydz}{|\bar{n}.\bar{i}|}$$

$\bar{F}.\bar{n}$= $(z\bar{\imath} + x\bar{\jmath} - 3y^2z\bar{k}).(\frac{x\,\bar{\imath}+y\,\bar{\jmath}}{4})$= $\frac{xz+xy}{4}$

$\bar{n}.\bar{\imath}$= $(\frac{x\,\bar{\imath}+y\,\bar{\jmath}}{4})$. i= $\frac{x}{4}$

For the surface $x^2 + y^2 = 16$ in the yz-plane , x=0 $=> y = 4$

Z limits are 0 to 5

Y limits are 0 to 4

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\bar{F}.\bar{n}\, dydz}{|\bar{n}.\bar{\imath}|}$$

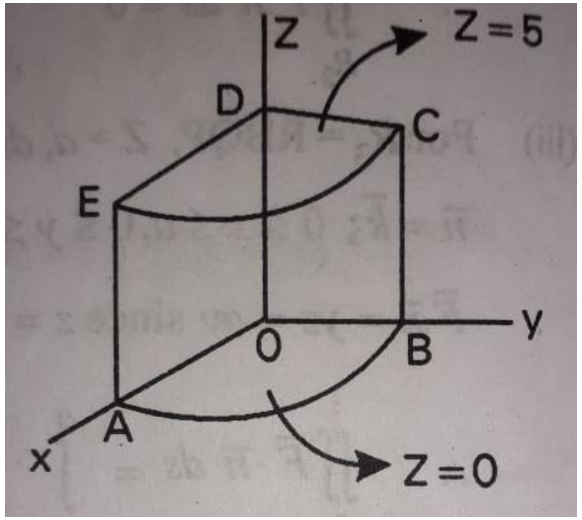
$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\frac{xz + xy^2}{4} dydz}{\left|\frac{x}{4}\right|}$$

$$= \int_{y=0}^{4} \int_{z=0}^{5} (y + z)\, dydz = \int_{y=0}^{4} \left[yz + \frac{z^2}{2}\right]_0^5 dy$$

$$= \int_{y=0}^{4} (5y + \frac{25}{2})\, dy = \left[5\frac{y^2}{2} + \frac{25y}{2}\right]_0^4$$

$$= 40 + 50 = 90$$

4) Evaluate $\int \bar{F}.\bar{n}\, dS$, where $\bar{F} = yz\bar{\imath} + 2xy^2\bar{\jmath} + xz^2\bar{k}$ and S is the curved surface $x^2 + y^2$ =9 included in the first octant between z=0 and z=2.



Sol: Given surface S is the curved surface ABCEA.

Let $\emptyset = x^2 + y^2 - 9$

Normal to the surface S is grad $\emptyset$

Normal vector = grad $\emptyset$= $\nabla\emptyset$ = $\bar{\imath}\frac{\partial\emptyset}{\partial x}+\bar{\jmath}\frac{\partial\emptyset}{\partial y}+\bar{k}\,\frac{\partial\emptyset}{\partial z}$

$$= 2x\,\bar{\imath} + 2y\,\bar{\jmath} + o\,\bar{k}$$

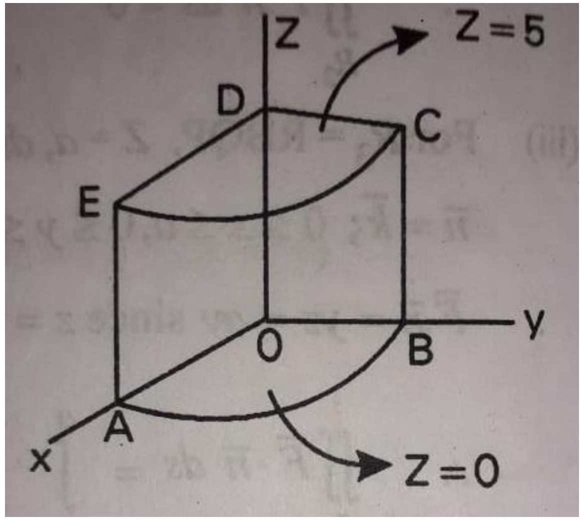
Unit normal vector is $\bar{n} = \frac{2x\,\bar{\imath}+2y\,\bar{\jmath}+o\,\bar{k}}{|2x\,\bar{\imath}+2y\,\bar{\jmath}+o\,\bar{k}|}=\frac{2x\,\bar{\imath}+2y\,\bar{\jmath}+o\,\bar{k}}{\sqrt{4x^2+4y^2+0^2}}$

$$=\frac{2(x\,\bar{\imath}+y\,\bar{\jmath})}{2\sqrt{9}} = \frac{x\,\bar{\imath}+y\,\bar{\jmath}}{3}$$

Consider the curved region R: OBCD

Let R be the projection of S on yz- plane . Then

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\bar{F}.\bar{n}\,dydz}{|\bar{n}.\bar{\imath}|}$$



$\bar{F}.\bar{n}$= $(yz\bar{\imath} + 2xy^2\bar{\jmath} + xz^2\bar{k}).(\frac{x\,\bar{\imath}+y\,\bar{\jmath}}{3})$= $\frac{xyz+2xy^3}{3}$

$\bar{n}.\bar{\imath}$= $(\frac{x\,\bar{\imath}+y\,\bar{\jmath}}{3}).\ \dot{i}$= $\frac{x}{3}$

For the surface $x^2 + y^2 = 9$ in the yz-plane , x=0 => y =3

Z limits are 0 to2

Y limits are 0 to 3

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\bar{F}.\bar{n}\ dydz}{|\bar{n}.\bar{\imath}|}$$

$$\int_S \bar{F}.\bar{n}dS = \iint_{R_2} \frac{\frac{xyz + 2xy^3}{3} dydz}{\left|\frac{x}{3}\right|}$$

$$= \int_{y=0}^{3} \int_{z=0}^{2} (yz + 2y^3)\ dydz = \int_{y=0}^{3} \left[y\frac{z^2}{2} + 2\frac{y^4}{4}\right]_0^2 dy$$

$$= \int_{y=0}^{3} (2y + 8)\ dy = \left[2\frac{y^2}{2} + 8y\right]_0^3$$

$$= 9 + 24 = 33$$

PRATICE PROBLEM

1. Evaluate $\int \bar{F}.\bar{n}\, dS$, where $\bar{F} = xz\bar{\imath} + 2y^2\bar{\jmath} + xy\bar{k}$ and S is the curved surface $x^2 + y^2 = 25$ included in the first octant between z=0 and z=5.

Volume Integrals

If $\overline{F}(r) = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a vector point function defined over volume V so that $dv = dxdydz$ then the volume integral is given by

$$\int_V \overline{F}dv = \iiint (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})dxdydz$$

$$= \bar{i}\iiint F_1 dxdydz + \bar{j}\iiint F_2 dxdydz + \bar{k}\iiint F_3 dxdydz$$

Problem:

If $\overline{F} = (x^2 - yz)\bar{i} + y\bar{j} + z\bar{k}$ then find the volume integral of over the region bounded by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$

Sol: Given, $\overline{F} = (x^2 - yz)\bar{i} + y\bar{j} + z\bar{k}$

$$\int_V \overline{F} dv = \bar{i} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} ((x^2 - yz)\bar{i} + y\bar{j} + z\bar{k}) dz dy dx$$

$$= \bar{i} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} (x^2 - yz) dz dy dx + \bar{j} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} y dz dy dx$$

$$+ \bar{k} \int_{x=0}^{a} \int_{y=0}^{b} \int_{z=0}^{c} z dz dy dx$$

$$= \bar{i} \int_{x=0}^{a} \int_{y=0}^{b} \left(x^2 z - y \frac{z^2}{2}\right)_{z=0}^{c} dydx + \bar{j} \int_{x=0}^{a} \int_{y=0}^{b} (yz)_{z=0}^{c} dydx$$

$$+ \bar{k} \int_{x=0}^{a} \int_{y=0}^{b} \left(\frac{z^2}{2}\right)_{z=0}^{c} dydx$$

$$= \bar{i} \int_{x=0}^{a} \int_{y=0}^{b} \left(x^2 c - y \frac{c^2}{2}\right) dydx + \bar{j} \int_{x=0}^{a} \int_{y=0}^{b} yc\, dydx + \bar{k} \int_{x=0}^{a} \int_{y=0}^{b} \frac{c^2}{2} dydx$$

$$= \bar{i}\int_{x=0}^{a}\left(cx^2y-\frac{c^2}{2}\frac{y^2}{2}\right)_{y=0}^{b}dx+\bar{j}\int_{x=0}^{a}\left(c\frac{y^2}{2}\right)_{y=0}^{b}dx$$

$$+\bar{k}\int_{x=0}^{a}\left(\frac{c^2}{2}y\right)_{y=0}^{b}dx$$

$$= \bar{i}\int_{x=0}^{a}\left(x^2bc-\frac{c^2}{2}\frac{b^2}{2}\right)dx+\bar{j}\int_{x=0}^{a}c\frac{b^2}{2}dx+\bar{k}\int_{x=0}^{a}\frac{c^2}{2}bdx$$

$$= \bar{i}\left(\frac{x^3}{3}bc - \frac{b^2c^2x}{4}\right)_{x=0}^{a} + \bar{j}\left(c\frac{b^2}{2}x\right)_{x=0}^{a}$$

$$+ \bar{k}\left(\frac{bc^2}{2}x\right)_{x=0}^{a}$$

$$= \left(\frac{a^3}{3}bc - \frac{ab^2c^2}{4}\right)\bar{i} + \frac{ab^2c}{2}\bar{j} + \frac{abc^2}{2}\bar{k}$$

Problem:

If $\overline{F} = 2xz\overline{i} - x\overline{j} + y^2\overline{k}$ then find the volume integral over the region bounded by $x = 0, x = 2, y = 0, y = 6, z = x^2, z = 4$

Sol: Given, $\overline{F} = 2xz\overline{i} - x\overline{j} + y^2\overline{k}$

$$\int_V \overline{F}dv = \overline{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}((2xz\overline{i} - x\overline{j} + y^2\overline{k})dzdydx$$

$$= \overline{i}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xzdzdydx - \overline{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} xdzdydx$$

$$+ \overline{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y^2dzdydx$$

$$= \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} \left(xz^2\right)_{z=x^2}^{4} dydx - \bar{j} \int_{x=0}^{2} \int_{y=0}^{6} \left(xz\right)_{z=x^2}^{4} dydx$$

$$+ \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} \left(y^2 z\right)_{z=x^2}^{4} dydx$$

$$= \bar{i} \int_{x=0}^{2} \int_{y=0}^{6} x(16 - x^4 c)dydx - \bar{j} \int_{x=0}^{2} \int_{y=0}^{6} x(4 - x^2)dydx$$

$$+ \bar{k} \int_{x=0}^{2} \int_{y=0}^{6} y^2(4 - x^2)dydx$$

$$= \bar{i}\int_{x=0}^{2}(16x - x^5)(y)_{y=0}^{6}dx - \bar{j}\int_{x=0}^{2}(4x - x^3)(y)_{y=0}^{6}dx$$

$$+\bar{k}\int_{x=0}^{2}(4 - x^2)\left(\frac{y^3}{3}\right)_{y=0}^{6}dx$$

$$= \bar{i}\int_{x=0}^{2}(16x - x^5)(6)dx - \bar{j}\int_{x=0}^{2}(4x - x^3)(6)dx$$

$$+\bar{k}\int_{x=0}^{2}(4 - x^2)\left(\frac{216}{3}\right)dx$$

$$= \bar{i}\left(8x^2 - \frac{x^6}{6}\right)_{x=0}^{2}(6) - \bar{j}\left(2x^2 - \frac{x^4}{4}\right)_{x=0}^{2}(6)$$

$$+ \bar{k}\left(4x - \frac{x^3}{3}\right)_{x=0}^{2}\left(\frac{216}{3}\right)$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}$$

Problem:

If $\overline{F} = xy\overline{i} + yz\overline{j} + zx\overline{k}$ then evaluate

$(i)\int_V \nabla \cdot \overline{F} dv$ $(ii)\int_V \nabla \times \overline{F} dv$ where V is region bounded by

$x = 0, x = 2, y = 0, y = 1, z = 0, z = x$

Sol: Given, $\overline{F} = xy\overline{i} + yz\overline{j} + zx\overline{k}$

$$(i)\int_V \nabla \cdot \overline{F} dv$$

$$\nabla \cdot \overline{F} = \frac{\partial \overline{F}}{\partial x} + \frac{\partial \overline{F}}{\partial y} + \frac{\partial \overline{F}}{\partial z} = y + z + x$$

Now,

$$\int_V \nabla \cdot \overline{F} dv = \int_{x=0}^{2} \int_{y=0}^{1} \int_{z=0}^{x} (x+y+z) dz dy dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{1} \left(xz + yz + \frac{z^2}{2}\right)_{z=0}^{x} dy dx$$

$$= \int_{x=0}^{2} \int_{y=0}^{1} \left(x^2 + yx + \frac{x^2}{2}\right) dy dx$$

$$= \int_{x=0}^{2} \left(x^2 y + x\frac{y^2}{2} + \frac{x^2}{2} y\right)_{y=0}^{1} dx$$

$$= \int_{x=0}^{2} \left(x^2 + \frac{x}{2} + \frac{x^2}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x^3}{6}\right)_0^2$$

$$= \left(\frac{2^3}{3} + \frac{2^2}{2} + \frac{2^3}{6}\right)$$

$$= \frac{8}{3} + 2 + \frac{4}{3}$$

$$= \frac{18}{3}$$

$$(ii)\int_V \nabla \times \overline{F}\,dv$$

$$\nabla \times \overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & zx \end{vmatrix}$$

$$= \vec{i}\,[0 - y] - \vec{j}\,[z - 0] + \vec{k}\,[0 - x]$$

$$= -y\vec{i} - z\vec{j} - x\vec{k}$$

Now, $$(ii)\int_V \nabla \times \bar{F}\,dv = -\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{x}(yi + zj + xk)\,dzdydx$$

$$= -\bar{i}\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{x} y\,dzdydx - \bar{j}\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{x} z\,dzdydx - \bar{k}\int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{x} x\,dzdydx$$

$$= -\bar{i}\int_{x=0}^{2}\int_{y=0}^{1}(yz)_{z=0}^{x}\,dydx - \bar{j}\int_{x=0}^{2}\int_{y=0}^{1}\left(\frac{z^2}{2}\right)_{z=0}^{x}dydx$$

$$-\bar{k}\int_{x=0}^{2}\int_{y=0}^{1}(xz)_{z=0}^{x}\,dydx$$

$$= -\bar{i}\int_{x=0}^{2}\int_{y=0}^{1}xy\,dydx - \bar{j}\int_{x=0}^{2}\int_{y=0}^{1}\frac{x^2}{2}dydx$$

$$-\bar{k}\int_{x=0}^{2}\int_{y=0}^{1}x^2\,dydx$$

$$= -\bar{i} \int_{x=0}^{2} x \left(\frac{y^2}{2}\right)_{y=0}^{1} dx - \bar{j} \int_{x=0}^{2} \frac{x^2}{2} (y)_{y=0}^{1} dx$$

$$- \bar{k} \int_{x=0}^{2} x^2 (y)_{y=0}^{1} dx$$

$$= -\bar{i} \int_{x=0}^{2} \frac{x}{2} dx - \bar{j} \int_{x=0}^{2} \frac{x^2}{2} dx - \bar{k} \int_{x=0}^{2} x^2 dx$$

$$= -\bar{i} \left(\frac{x^2}{4}\right)_{x=0}^{2} - \bar{j} \left(\frac{x^3}{6}\right)_{x=0}^{2} - \bar{k} \left(\frac{x^3}{3}\right)_{x=0}^{2}$$

$$= -\bar{i}\,\frac{2^2}{4} - \bar{j}\,\frac{2^3}{6} - \bar{k}\,\frac{2^3}{3}$$

$$= -\bar{i}\,\frac{4}{4} - \bar{j}\,\frac{8}{6} - \bar{k}\,\frac{8}{3}$$

$$= -\bar{i} - \frac{4}{3}\,\bar{j} - \frac{8}{3}\,\bar{k}$$

Problem:

If $\phi = 2x$ then evaluate $\iiint_V \phi\, dv$ where V is region bounded by

$$x = 0, y = 0, z = 0, 2x + 2y + z = 4$$

Sol: Given,

$$\phi = 2x$$

The limits are:

$z = 0$ to $z = 4 - 2x - 2y$

$y = 0$ to $y = (4 - 2x)/2$

$x = 0$ to $x = 4/2 = 2$

Now,

$$\iiint_V \phi dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}(2xz)_{z=0}^{4-2x-2y}dydx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}2x(4-2x-2y)dydx = 4\int_{x=0}^{2}\int_{y=0}^{2-x}(2x-x^2-xy)dydx$$

$$= 4\int_{x=0}^{2}\left(2xy+x^2y-\frac{xy^2}{2}\right)_{y=0}^{2-x}dx$$

$$= 2\int_{x=0}^{2}(x^3-4x^2+4x)dx$$

$$=2\left(\frac{x^4}{4}-4\frac{x^3}{3}+4\frac{x^2}{2}\right)_0^2$$

$$=\left(\frac{x^4}{2}-8\frac{x^3}{3}+4x^2\right)_0^2$$

$$=\frac{16}{2}-8\frac{(8)}{3}+4(4)$$

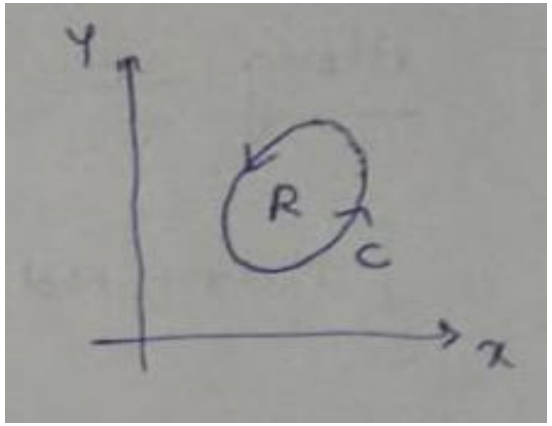
$$=-8$$

Practice Problem

Evaluate $\int_V \overline{F} dv$ when $\overline{F} = x\overline{i} + y\overline{j} + z\overline{k}$ where V is the region bounded by $x = 0, x = 2y = 0, y = 1, z = 0, z = 4$

Green's Theorem:-

If R is a closed region in XY- plane bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivatives in R, then $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$



1) Verify Green's theorem in the plane for $\int_c (x^2 - xy^3)\mathrm{dx} + (\mathrm{y}^2 - 2\mathrm{xy})\mathrm{dy}$ where c is a square with vertices (0,0), (2,0), (2,2), (0,2).

Sol:)

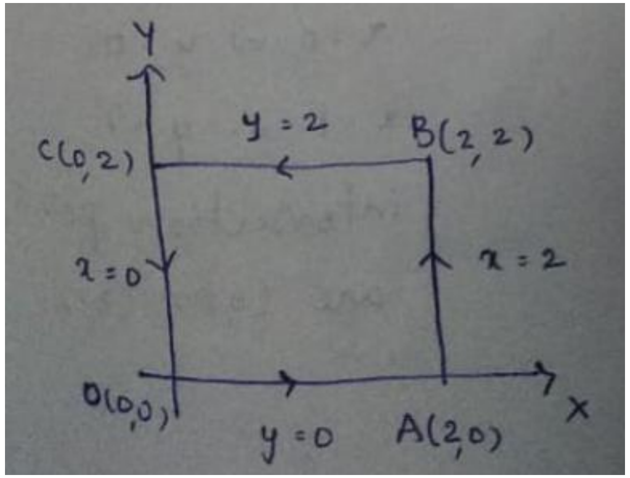
Green's theorem is

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Here M= $x^2 - xy^3$ N= $y^2 - 2xy$

L.H.S:-

$$\int_c M dx + N dy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$



Along OA: y=0 =>dy = 0, x : 0 → 2

$$\int_{OA} M\,dx + N\,dy = \int_0^2 x^2\,dx = \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}.$$

Along AB: x=2 =>dx = 0, y : 0 → 2

$$\int_{AB} M\,dx + N\,dy = \int_0^2 (y^2 - 4y)dy$$

$$=\left(\frac{y^3}{3}-\frac{4y^2}{2}\right)_0^2=\frac{8}{3}-8=\frac{-16}{3}$$

Along BC: y=2 => dy=0 x : 2 → 0

$$\int_{BC} M\,dx+N\,dy=\int_2^0 (x^2-8x)\,dx$$

$$=\left(\frac{x^3}{3}-\frac{8x^2}{2}\right)_2^0=\frac{-8}{3}+16=\frac{40}{3}$$

Along CO: x=0 => dx = 0 y : 2 → 0

$$\int_{CO} M\,dx+N\,dy=\int_2^0 y^2\,dy$$

$$=\left(\frac{y^3}{3}-\frac{4y^2}{2}\right)_0^2=\frac{8}{3}-8=\frac{-16}{3}$$

Along BC: y=2 => dy=0 x : 2 → 0

$$\int_{BC} M\,dx+N\,dy=\int_2^0 (x^2-8x)\,dx$$

$$=\left(\frac{x^3}{3}-\frac{8x^2}{2}\right)_2^0=\frac{-8}{3}+16=\frac{40}{3}$$

Along CO: x=0 => dx = 0 y : 2 → 0

$$\int_{CO} M\,dx+N\,dy=\int_2^0 y^2\,dy$$

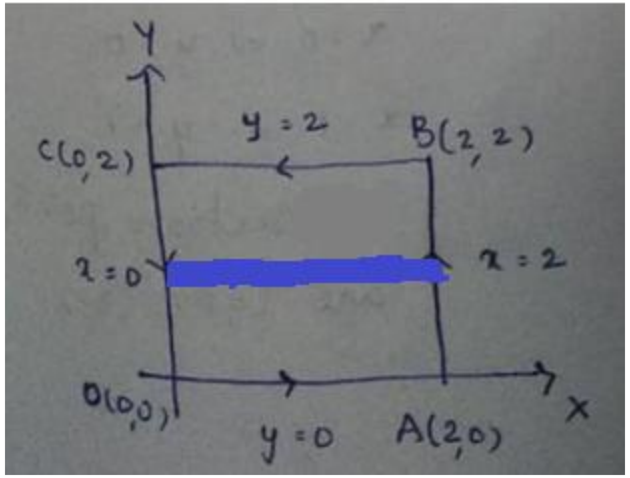
$$=\left(\frac{y^3}{3}\right)_0^2 = \frac{-8}{3}$$

$$\int_c Mdx + Ndy = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \frac{8}{3} - \frac{16}{3} + \frac{40}{3} - \frac{8}{3}$$

$$= 8$$

R.H.S:- $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$



M= $x^2 - xy^3$ N= $y^2 - 2xy$

$$\frac{\partial M}{\partial y} = -3xy^2 \qquad \frac{\partial N}{\partial x} = -2y$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{x=0}^{2} \int_{y=0}^{2} (-2y + 3xy^2) dxdy$$

$$= \int_0^2 \left[\frac{-2y^2}{2} + \frac{3xy^3}{3}\right]_0^2 dx$$

$$=\int_0^2 [-y^2 + xy^3]_0^2 \, dx$$

$$=\int_0^2 (-4 + 8x)dx$$

$$= \left[-4x + \frac{8x^2}{2}\right]_0^2 = -8+16 = 8$$

L.H.S=R.H.S

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

Hence Green's theorem is verified..

2) Verify Green's theorem for $\int_c (xy + y^2)\mathrm{dx} + \mathrm{x}^2\mathrm{dy}$ where c is Bounded by y=x and y=x^2

Sol:)

Green's theorem is

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

L.H.S:

$$\int_c Mdx + Ndy = \int_{c_1} + \int_{c_2}$$

$y = x \qquad y = x^2$

$x = x^2$

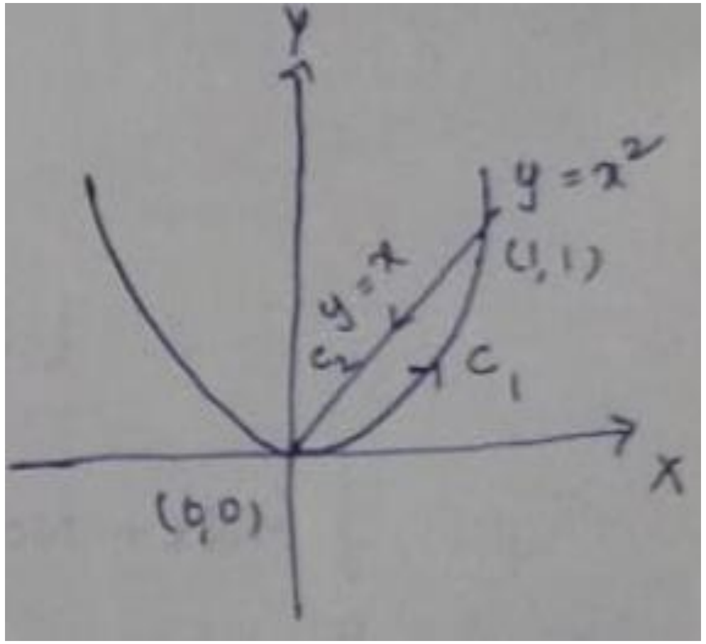
$x^2 - x = 0$

$x(1-x)=0$

$x = 0 \, , \, x = 1$

$x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$



Intersection points are (0,0) , (1,1)

Along c_1:

$y=x^2 \Rightarrow dy = 2x\,dx$

$x : 0 \to 1$

$$\int_{C_1} M\,dx + N\,dy = \int_0^1 (x^3 + x^4)dx + x^2(2xdx)$$

$$= \int_0^1 (\,3x3 + x4)dx = \left(\frac{3x^4}{4} - \frac{x^5}{5}\right)_0^1$$

$$= \frac{3}{4} + \frac{1}{5} = \frac{19}{20}.$$

Along c_2:

$y = x \Rightarrow dy = dx \qquad x : 1 \rightarrow 0$

$$\int_{C_2} M\,dx + N\,dy = \int_1^0 (x^2 + x^2)dx + x^2 dx$$

$$= \int_1^0 3x^2 dx = 3\left(\frac{3x^3}{3}\right)_1^0 = -1$$

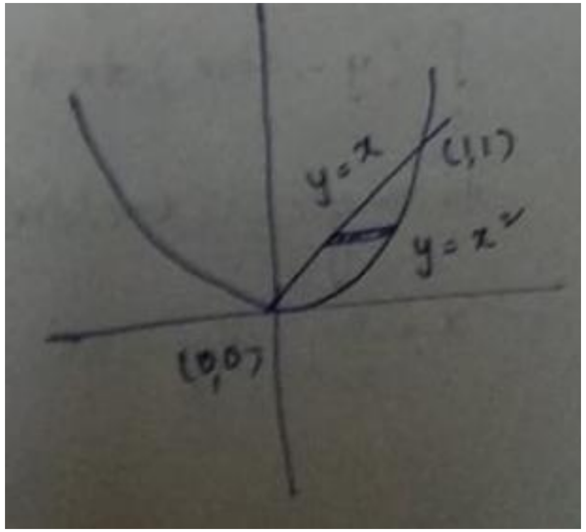
$$\int_c Mdx + Ndy = \int_{c_1} + \int_{c_2}$$

$$= \frac{19}{20} - 1 = \frac{-1}{20}$$

R.H.S: $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$

M= xy+y^2 N= x^2

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$$



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} (x - 2y)\, dxdy$$

$$= \int_{y=0}^{1} \left[\int_{y}^{\sqrt{y}} (x - 2y)\, dx\right] dy$$

$$= \int_0^1 \left[\frac{x^2}{2} - 2xy\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2\sqrt{y}y - \frac{y^2}{2} + 2y^2\right]_y^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\frac{y}{2} - 2y^{\frac{3}{2}} + \frac{3y^2}{2}\right]_y^{\sqrt{y}} dy$$

$$= \left[\frac{y^2}{2} - 2y^{\frac{5}{2}} . \frac{2}{5} + \frac{3}{2}\frac{y^3}{3}\right]_0^1$$

$$=\frac{1}{4}-\frac{4}{5}+\frac{1}{2}$$

$$=\frac{-1}{20}$$

L.H.S = R.H.S

Hence Green's theorem is verified..

3) Evaluate Green's Theorem $\int_c (y - \sin x)\mathrm{dx} + \cos x \,\mathrm{dy}$ where c is the triangle enclosed by the lines y=0, x=$\frac{\pi}{2}$,

$\pi y = 2x$

Sol:)

Let M= y-$\sin x$ N= $\cos x$

$\frac{\partial M}{\partial y} = 1$ $\frac{\partial y}{\partial x} = -\sin x$

By Green's theorem

$$\int_c Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

$$\int_c (y - \sin x)\, dx + \cos x\, dy = \int_{y=0}^{1} \int_{x=\frac{\pi y}{2}}^{\frac{\pi}{2}} (-\sin x - 1) dx dy$$

$$= \int_{y=0}^{1} \left[\int_{\frac{\pi y}{2}}^{\frac{\pi}{2}} (-\sin x - 1) dx \right] dy$$

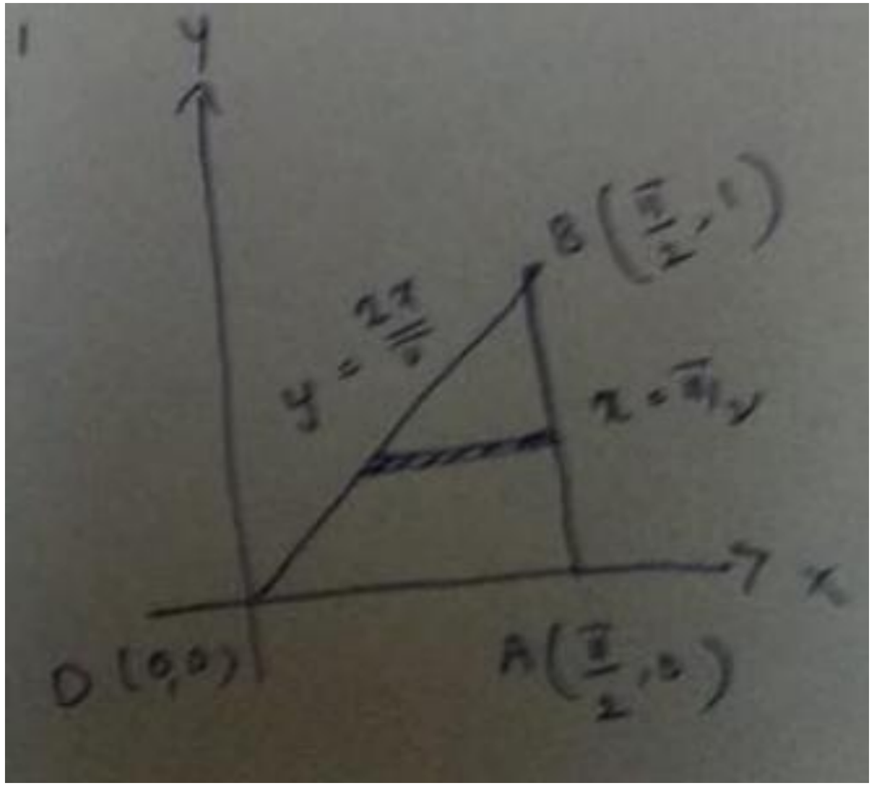
$$= \int_0^1 [\cos x - x]_{\frac{\pi y}{2}}^{\frac{\pi}{2}}\, dy$$

$$= \int_0^1 \left[-\frac{\pi}{2} - \cos\frac{\pi y}{2} + \frac{\pi y}{2} \right] dy$$

$$= \left[-\frac{\pi y}{2} - \sin\frac{\pi y}{2}.\frac{2}{\pi} + \frac{\pi y^2}{4}\right]_0^1$$

$$= \frac{-\pi}{2} - \frac{2}{\pi} + \frac{\pi}{4}$$

$$= \frac{-\pi}{4} - \frac{2}{\pi}$$



Gauss divergence theorem:

Let S be a closed surface enclosing a volume V. If $\bar{F}$ is a continuously differentiable vector point function, then

$$\int_{V} div\bar{F}\,dv = \int_{S} \bar{F}.\bar{n}\,dS$$

Where $\bar{n}$ is the unit outward normal vector

1) Evaluate $\int_S \bar{F}.\bar{n}ds$, if $F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane x+ y+ z =1.

Sol: From Gauss divergence theorem

$$\int_V div\bar{F}dv = \int_S \bar{F}.\bar{n}ds$$

Given, $F = xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k}$

$div\bar{F} = \nabla.\bar{F}$

$$= (\bar{\imath}\frac{\partial}{\partial x} + \bar{\jmath}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}).(xy\bar{\imath} + z^2\bar{\jmath} + 2yz\bar{k})$$

$$= \frac{\partial(xy)}{\partial x} + \frac{\partial z^2}{\partial y} + 2\frac{\partial(yz)}{\partial z}$$

= y+ 0+ 2y = 3y

Given curve is the tetrahedron bounded by x=0, y=0, z=0 and the plane x+ y+ z =1

z limits 0 to 1- x – y

Y limits 0 to 1-x

X limits 0 to 1

Hence $\int_S \bar{F}.\bar{n}ds = \int_V div\bar{F}dv$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} 3y\ dx\ dy\ dz$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} (3yz)_{z=0}^{1-x-y} dydx$$

.

$$= \int_{x=0}^{1}\int_{y=0}^{1-x} 3y(1-x-y)\,dy\,dx = \int_{x=0}^{1}\int_{y=0}^{1-x} (3y-3xy-3y^{2})\,dy\,dx$$

$$= \int_{x=0}^{1} \left[3\frac{y^{2}}{2} - 3x\frac{y^{2}}{2} - 3\frac{y3}{3}\right]_{y=0}^{1-x} dx$$

$$= \int_{x=0}^{1} \left[3\frac{(1-x)^{2}}{2} - 3x\frac{(1-x)^{2}}{2} - 3\frac{(1-x)3}{3}\right] dx$$

$$= 3\int_{x=0}^{1} \left[\frac{(1-x)^{2}}{2}(1-x) - \frac{(1-x)3}{3}\right] dx$$

•

$$= 3\int_{x=0}^{1}\left[\frac{(1-x)^3}{2}-\frac{(1-x)3}{3}\right]dx$$

$$= 3\int_{x=0}^{1}\left[\frac{(1-x)^3}{6}\right]dx$$

$$= 3\left[-\frac{(1-x)^4}{6(4)}\right]_0^1$$

$$= \frac{1}{8}$$

2) Verify Gauss divergence theorem for $\bar{F} = 4xz\bar{\imath} - y^2\bar{\jmath} + yz\bar{k}$, taken over the surface of the cube bounded by the planes x=0, x=a, y=0, y=a, z=0, z=a.

Sol: From Gauss divergence theorem

$$\int_V div\bar{F}dv = \int_S \bar{F}.\bar{n}ds$$

Given, $F = 4xz\bar{\imath} - y^2\bar{\jmath} + yz\bar{k}$

$div\bar{F} = \nabla.\bar{F}$

$= (\bar{\imath}\frac{\partial}{\partial x}+\bar{\jmath}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}).(4xz\bar{\imath} - y^2\bar{\jmath} + yz\bar{k})$

$= 4\frac{\partial(xz)}{\partial x} - \frac{\partial y^2}{\partial y}+\frac{\partial(yz)}{\partial z}$

= 4z-2y+ y = 4z-y

Given surface is of the cube bounded by the planes x=0, x=a, y=0, y=a, z=0, z=a.

z limits 0 to a

Y limits 0 to a

X limits 0 to a

Hence $\int_S \bar{F}.\bar{n}dS = \int_V div\bar{F}dv$

$$= \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} (4z - y)dzdydx$$

$$= \int_{x=0}^{a} \int_{y=0}^{a} \left[4.\frac{z^2}{2} - yz\right]_{z=0}^{a} dydx$$

.

$$= \int_{x=0}^{a} \int_{y=0}^{a} [4.\frac{a^2}{2} - ay] dy dx$$

$$= \int_{x=0}^{a} [4.\frac{a^2}{2} y - a\frac{y^2}{2}]_{y=0}^{a} dx$$

$$= \int_{x=0}^{a} [4.\frac{a^2}{2}.a - a.\frac{a^2}{2}] dx$$

$$= [4.\frac{a^3}{2} x - \frac{a^3}{2} x]_{x=0}^{a}$$

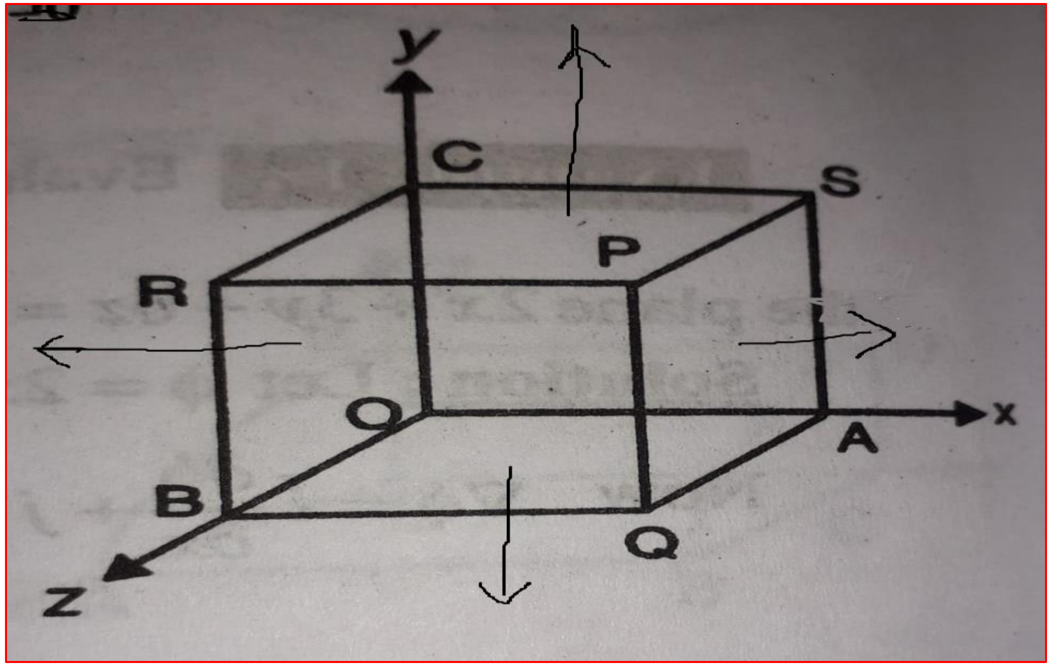
$$= [2a^4 - \frac{a^4}{2}] = \frac{3a^4}{2}$$

Verification:

consider the volume within the cube PQASCRBO in figure bounded by x=0, x=a, y=0, y=a, z=0, z=a.

Here $\bar{F} = 4xz\bar{\imath} - y^2\bar{\jmath} + yz\bar{k}$

Let us calculate $\int_S \bar{F}.\bar{n}\, dS$ for each face of the cube.

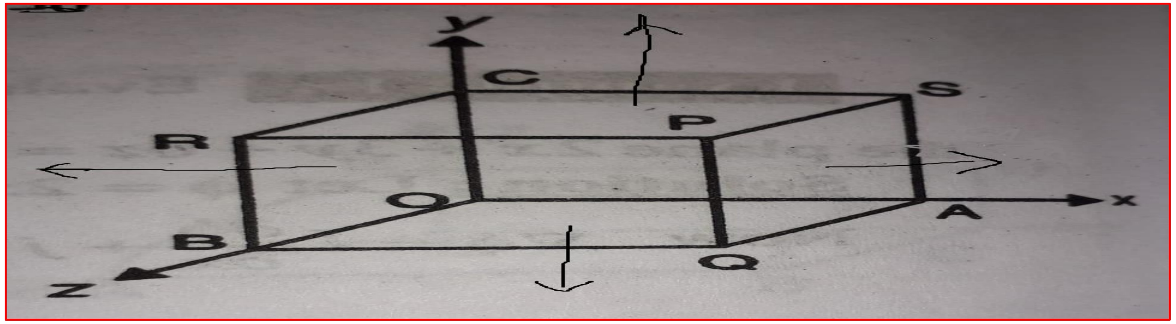


$$\bar{F} = 4xz\bar{\imath} - y^2\bar{\jmath} + yz\bar{k}$$

I) Along the face R_1 =OCRB, it is in yz-plane

X=0, ds= dydz, $\bar{n} = -\bar{\imath}$

$0 \le y \le a, 0 \le z \le a$



$\bar{F}.\bar{n} = -4xz = 0 \ (since\ x = 0)$

$$\iint_{R_1} \bar{F}.\bar{n}\, dS = 0$$

II)Along the face R_2 = ASPQ, it is in yz-plane

X=a, ds= dydz, $\bar{n} = \bar{\imath}$

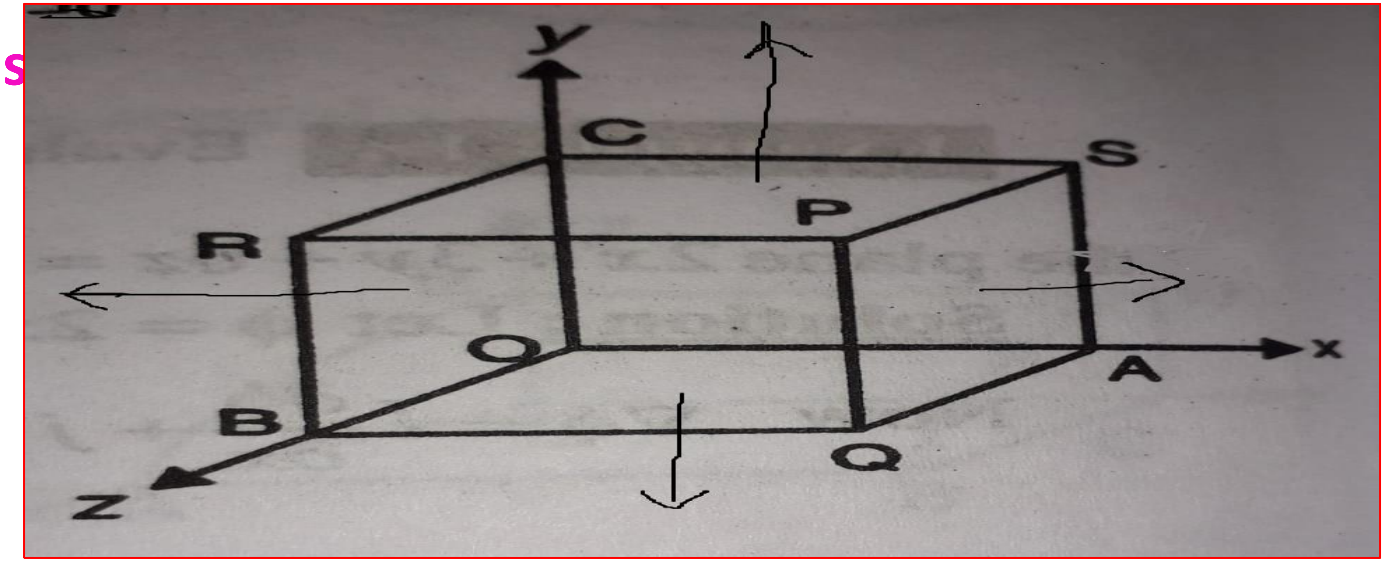
$0 \le y \le a, 0 \le z \le a$

$\bar{F}.\bar{n} = 4xz = 4az \quad (since\ x = a)$

$$\iint_{R_2} \bar{F}.\bar{n}\, dS = \int_{y=0}^{a}\int_{z=0}^{a} 4az\, dydz = 4a\int_{y=0}^{a}\left[\frac{z^2}{2}\right]_0^a dy = 4a.\frac{a^2}{2}\ .\ [y]_0^a = 2a^4$$

$\bar{F} = 4xz\bar{i} - y^2\bar{j} + yz\bar{k}$

III) Along the face R_3 =OAQB, it is



y=0, ds= dxdz, $\bar{n} = -\bar{j}$

$0 \le x \le a, 0 \le z \le a$

$\bar{F}.\bar{n} = y^2 = 0 \ (since\ y = 0)$

$$\iint_{R_3} \bar{F}.\bar{n}\, dS = 0$$

IV)Along the face R_4 = CSPR, it is in xz-plane

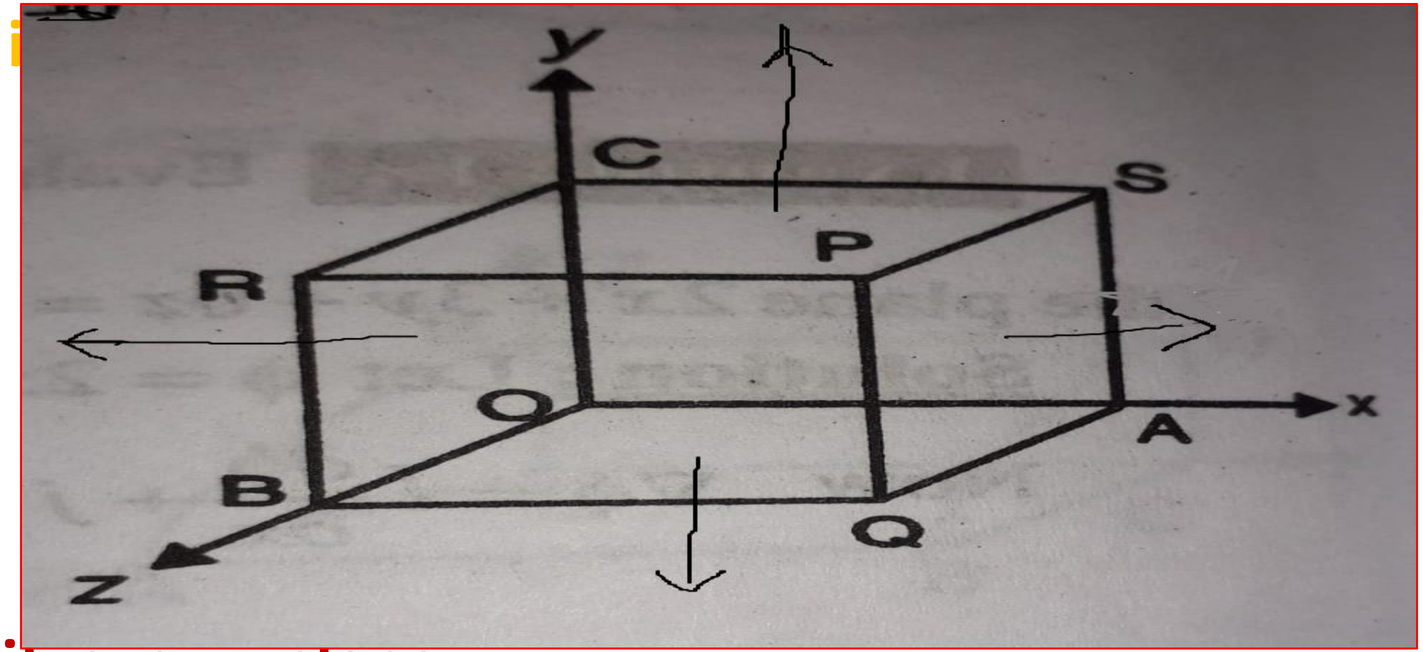
y=a, ds= dxdz, $\bar{n} = \bar{j}$

$0 \le x \le a, 0 \le z \le a$

$\bar{F}.\bar{n} = -y^2 = -a^2(since\ y = a)$

$$\iint_{R_4} \bar{F}.\bar{n}\, dS = \int_{x=0}^{a}\int_{z=0}^{a} (-a^2)dxdz = -a^2 \int_{x=0}^{a} [z]_0^a dx = -a^2.a \ [x]_0^a = -a^4$$

$$\bar{F} = 4xz\bar{\imath} - y^2\bar{\jmath} + yz\bar{k}$$



V) Along the face R_5 =OASC, it i

z=0, ds= dxdy, $\bar{n} = -\bar{k}$

$0 \le x \le a, 0 \le y \le a$

$$\bar{F}.\bar{n} = -yz = 0 \; (since\; z = 0)$$

$$\iint_{R_5} \bar{F}.\bar{n}\, dS = 0$$

VI)Along the face R_6 = PQBR, it is in xy-plane

z=a, ds= dxdy, $\bar{n} = \bar{k}$

$0 \le x \le a, 0 \le y \le a$

$$\bar{F}.\bar{n} = yz = ay \quad (since\; z = a)$$

$$\iint_{R_6} \bar{F}.\bar{n}\, dS = \int_{x=0}^{a}\int_{y=0}^{a} ay\, dxdy = a\int_{x=0}^{a}\left[\frac{y^2}{2}\right]_0^a dx = a.\frac{a^2}{2}\;.\;[x]_0^a = \frac{a^4}{2}$$

$$\int \bar{F}.\bar{n}\, dS = \iint_{R_1} \bar{F}.\bar{n}\, dS + \iint_{R_2} \bar{F}.\bar{n}\, dS + \iint_{R_3} \bar{F}.\bar{n}\, dS + \iint_{R_4} \bar{F}.\bar{n}\, dS + \iint_{R_5} \bar{F}.\bar{n}\, dS + \iint_{R_6} \bar{F}.\bar{n}\, dS$$

$$\int \bar{F}.\bar{n}\, dS = 0 + 2a^4 + 0 - a^4 + 0 + \frac{a^4}{2}$$

$$= \frac{3a^4}{2}$$

3) Evaluate $\int_S \bar{F}.\bar{n}ds$, if $F = x^2\bar{\imath} + z^2\bar{\jmath} + y\bar{k}$ over the tetrahedron bounded by x=0, y=0, z=0 and the plane 2x+ 2y+ z =4.

Sol: From Gauss divergence theorem

$$\int_V div\bar{F}dv = \int_S \bar{F}.\bar{n}ds$$

Given, $F = x^2\bar{\imath} + z^2\bar{\jmath} + y\bar{k}$

$div\bar{F} = \nabla.\bar{F}$

$$= \left(\bar{\imath}\frac{\partial}{\partial x}+\bar{\jmath}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right).(x^2\bar{\imath} + z^2\bar{\jmath} + y\bar{k})$$

$$= \frac{\partial X^2}{\partial x}+\frac{\partial z^2}{\partial y}+\frac{\partial y}{\partial z}$$

= 2x+ 0+ 0 = 2x

Given surface is the tetrahedron bounded by x=0, y=0, z=0 and the plane 2x+ 2y+ z =4

The limits are:

$z = 0$ to $z = 4 - 2x - 2y$

$y = 0$ to $y = (4 - 2x)/2$

$x = 0$ to $x = 4/2 = 2$

.

$$\iiint_V divF dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dz\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}(2xz)_{z=0}^{4-2x-2y}\,dy\,dx$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}2x(4-2x-2y)\,dy\,dx = 4\int_{x=0}^{2}\int_{y=0}^{2-x}(2x-x^2-xy)\,dy\,dx$$

$$= 4\int_{x=0}^{2}\left(2xy+x^2y-\frac{xy^2}{2}\right)_{y=0}^{2-x}dx$$

$$= 2\int_{x=0}^{2}(x^3-4x^2+4x)\,dx$$

•

$$=2\left(\frac{x^4}{4}-4\frac{x^3}{3}+4\frac{x^2}{2}\right)_0^2$$

$$=\left(\frac{x^4}{2}-8\frac{x^3}{3}+4x^2\right)_0^2$$

$$=\frac{16}{2}-8\frac{(8)}{3}+4(4)$$

$$=-8$$

4) Apply Gauss divergence theorem, prove that $\int \bar{r}.\bar{n}ds$ =3v.

Sol:

Let $\bar{r} = x\,\bar{\imath} + y\,\bar{j} + z\,\bar{k}$ and we know that div $\bar{r} = 3$

From Gauss divergence theorem,

$$\int_V div\bar{F}dv = \int_S \bar{F}.\bar{n}ds$$

Hence, $\int \bar{r}.\bar{n}ds$= $\int_V div\bar{r}dv$

=$\int_V 3dv$= 3V

PRATICE PROBLEM

1.Using Gauss divergence theorem, Evaluate $\int \bar{F}.\bar{n}\, dS$, where $\bar{F} = xz\bar{\imath} + 2y^2\bar{\jmath} + xy\bar{k}$ and S is the surface $x^2 + y^2 = 25$ included in the first octant between z=0 and z=5.

Stoke's Theorem

If S is a open surface bounded by a closed curve C and $\overline{F}$ is any differentiable vector point function then

$$\oint_C \overline{F} \cdot d\overline{r} = \int_S curl\overline{F}.\overline{n}ds$$

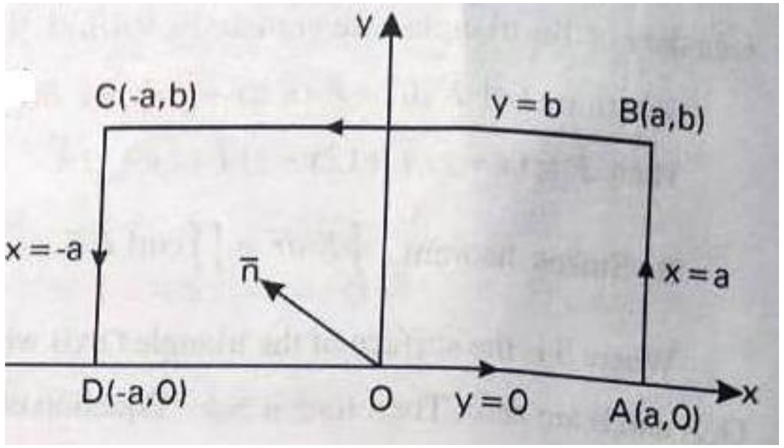
where C is traversed in the positive direction and $\overline{n}$ is unit outward drawn normal at any point in the surface

Problem

Verify Stoke's theorem for $\overline{F} = (x^2 + y^2)\overline{i} - 2xy\overline{j}$ taken round the rectangle bounded by the lines $x = \pm a, y = 0, y = b$

Sol: Let ABCD be a rectangle formed by the lines

$x = \pm a, y = 0, y = b$



By Stoke's theorem,

$$\oint_C \overline{F} \cdot d\overline{r} = \int_S curl\overline{F}.\overline{n}ds$$

Given $\overline{F} = (x^2 + y^2)\overline{i} - 2xy\overline{j}$

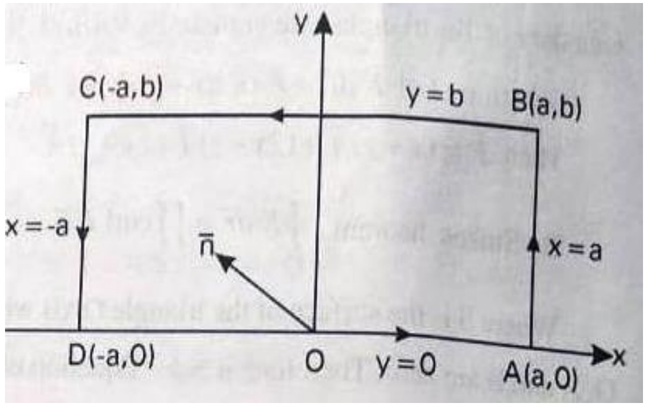
Given, $\overline{F} = (x^2 + y^2)\bar{i} - 2xy\bar{j}$

Consider L.H.S

$$\oint_C \overline{F} \cdot d\bar{r} = \oint_c \{(x^2 + y^2)\bar{i} - 2xy\bar{j}\} \cdot \{\bar{i}dx + \bar{j}dy\}$$

$$= \oint_c (x^2 + y^2)dx - 2xydy$$

$$= \int_{AB} + \int_{BC} + \int_{CA} + \int_{DA} \rightarrow (1)$$

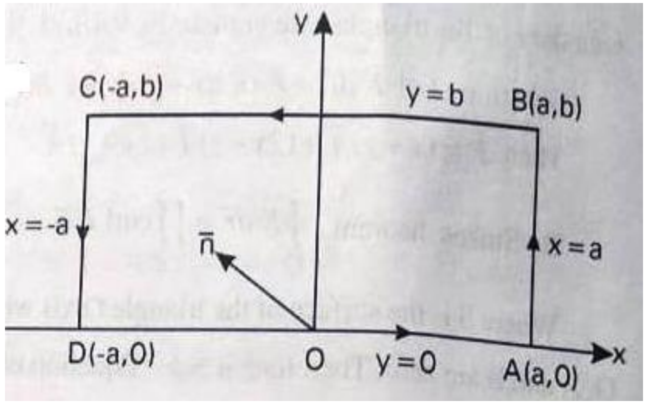


(i) Along AB, x=a,dx=0

From (1),

$$\int_{AB} \overline{F} \cdot d\bar{r} = \int_{y=0}^{b} -2aydy$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b = -ab^2$$



(ii) Along BC, y=b,dy=0

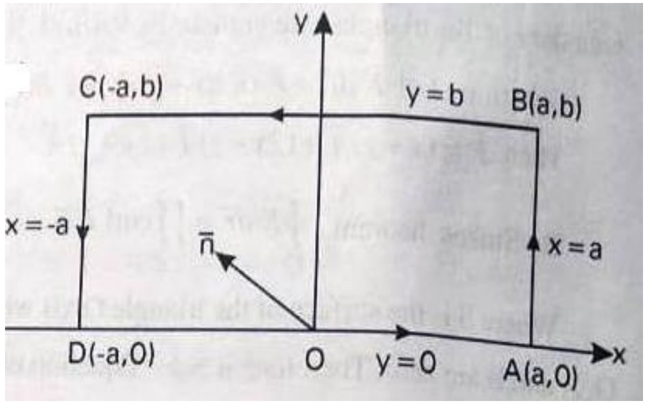
From (1),

$$\int_{BC} \overline{F}\cdot d\overline{r} = \int_{x=a}^{-a} (x^2 + b^2)dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^{-a} = -\frac{2a^3}{3} - 2ab^2$$

(iii) Along CD, x=-a,dx=0

From (1),



$$\int_{CD} \overline{F} \cdot d\overline{r} = \int_{y=b}^{0} 2ay\,dy$$

$$= a\left[y^2\right]_b^0 = -ab^2$$

(iv) Along DA, y=o,dy=0

From (1),

$$\int_{CD} \overline{F} \cdot d\overline{r} = \int_{x=-a}^{a} x^2 dx = \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

$$\therefore \oint_c \overline{F} \cdot d\overline{r} = \int_{AB} + \int_{BC} + \int_{CA} + \int_{DA}$$

$$= -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3}$$

$$= -4ab^2$$

Consider,

$$R.H.S = \int_S curl\overline{F} \cdot \overline{n}\, ds$$

$$curl\overline{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2 + y^2) & -2xy & 0 \end{vmatrix} = -4y\overline{k}$$

Since the rectangle is in xy-plane

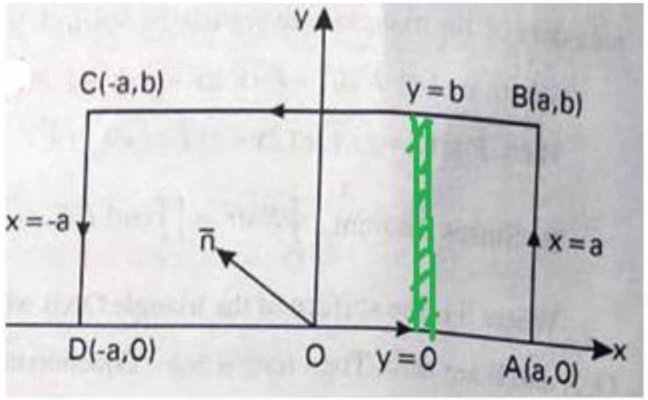
$$\bar{n} = \bar{k}, ds = dxdy$$

$$\int_S curl\bar{F} \cdot \bar{n} ds = \int_S -4y\bar{k} \cdot \bar{k} dxdy$$

$$= \int_{x=-a}^{a} \int_{y=0}^{b} -4y dxdy$$

$$= -4 \int_{y=0}^{b} y[x]_{-a}^{a} dy = -4 \int_{y=0}^{b} 2ay dy$$

$$= -4a[y^2]_0^b = -4ab^2$$



L.H.S=R.H.S

Hence Stoke's theorem is verified

Problem

Verify Stoke's theorem for $\overline{F} = (2x - y)\overline{i} - yz^2\overline{j} - y^2z\overline{k}$ over the upper half of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the projection in xy-plane

Sol: The boundary C of S is the circle in xy-plane

$$x^2 + y^2 = 1, z = 0$$

By Stoke's theorem,

$$\oint_C \overline{F} \cdot d\overline{r} = \int_S curl\overline{F}.\overline{n}ds$$

put $x = \cos\theta, y = \sin\theta, \theta : 0 \to 2\pi$

$$dx = -\sin\theta d\theta, dy = \cos\theta d\theta$$

Consider L.H.S

$$\oint_C \overline{F} \cdot d\overline{r} = \oint_c \{(2x - y^2)\overline{i} - yz^2\overline{j} - y^2 z\overline{k}\} \cdot \{\overline{i}dx + \overline{j}dy + \overline{k}dz\}$$

$$= \oint_c (2x - y)dx - yz^2 dy - y^2 z dz$$

$$= \oint_c (2x - y)dx (\because z = 0, dz = 0)$$

),

$$= -\int_0^{2\pi} (2\cos\theta - \sin\theta)\sin\theta d\theta$$

Consider L.H.S

$$= \int_0^{2\pi} \sin^2\theta \, d\theta - \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= \int_0^{2\pi} \frac{1-\cos 2\theta}{2} \, d\theta - \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + \frac{1}{2}\cos 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{2}(2\pi - 0) + 0 + \frac{1}{2}(\cos 4\pi - \cos 0) = \pi$$

Consider,

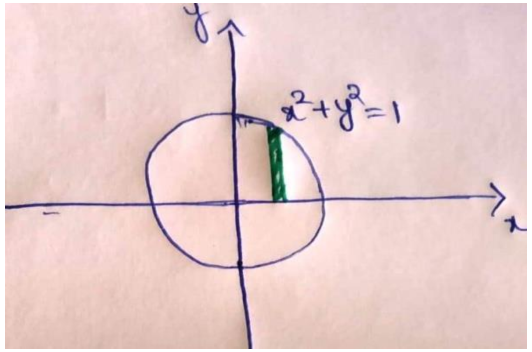
$$R.H.S = \int_S curl\overline{F} \cdot \overline{n}\, ds$$

$$curl\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix} = \overline{k}$$

$$\bar{n} = \bar{k}, ds = dxdy$$

$$\int_S curl\bar{F} \cdot \bar{n} ds = \int_S \bar{k} \cdot \bar{k} dxdy = \iint_R dxdy$$



$$= 4 \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} dydx$$

$$= 4 \int_{x=0}^{1} [y]_0^{\sqrt{1-x^2}} dx = 4 \int_{x=0}^{1} \sqrt{1-x^2} dx$$

$$= 4[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x]_0^1$$

$$= 4[\frac{1}{2}\sin^{-1}(1)] = 2\frac{\pi}{2} = \pi$$

Problem

Evaluate by Stoke's theorem $\int_c (e^x dx + 2ydy - dz)$ where C is the curve $x^2 + y^2 = 9, z = 2$

Sol: we have,

$$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$$

$$\bar{F} \cdot d\bar{r} = (e^x dx + 2ydy - dz)$$

Then

$$\bar{F} = e^x\bar{i} + 2y\bar{j} - \bar{k}$$

By Stokes's theorem,

$$\int_C \bar{F} \cdot d\bar{r} = \int_S curl\bar{F} . \bar{n} ds$$

Consider,

$$\int_S curl\overline{F} \cdot \overline{n}\, ds$$

$$curl\overline{F} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = \overline{i}(0-0) - \overline{j}(0-0) + \overline{k}(0-0)$$

$$= \overline{0}$$

Hence

$$\int_C \overline{F} \cdot d\overline{r} = \int_S curl\overline{F} . \overline{n} ds = 0$$

$$\therefore \int_c (e^x dx + 2ydy - dz) = \int_C \overline{F} \cdot d\overline{r} = 0$$

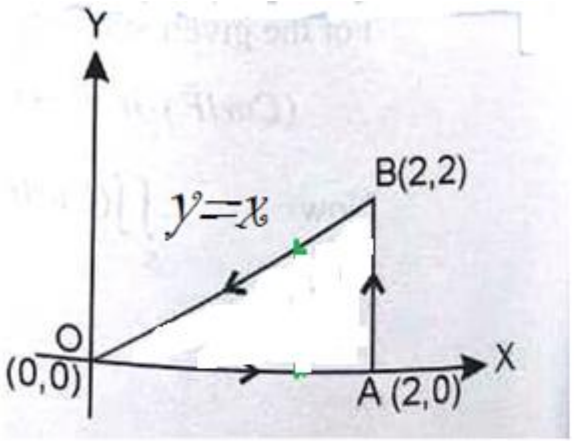
Problem

Evaluate by Stoke's theorem $\int_C \overline{F} \cdot d\overline{r}$ where $\overline{F} = 2y^2\overline{i} + 3x^2\overline{j} - (2x+z)\overline{k}$ and C is the boundaryof the trainglewhose vertices are (0,0,0), (2,0,0), (2,2,0)

Sol: Since z-coordinate of each vertex is zero , the triangle lies in xy-plane

By Stokes's theorem,

$$\int_C \overline{F} \cdot d\overline{r} = \int_S curl\overline{F}.\overline{n}ds$$



Consider, $\int_S curl\overline{F}\cdot\overline{n}\,ds$

$$curl\overline{F}=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$$

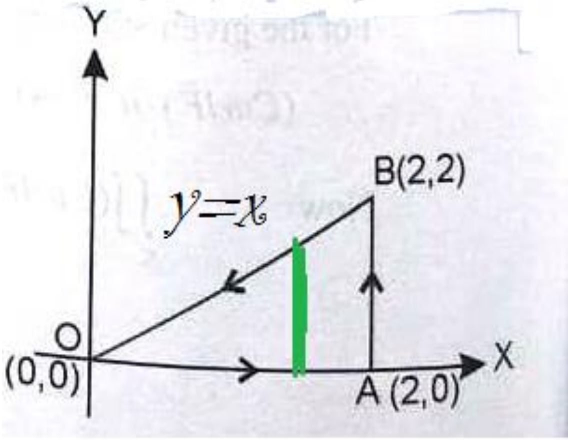
$$=\begin{vmatrix}\overline{i} & \overline{j} & \overline{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y^2 & 3x^2 & -2x-z\end{vmatrix}$$

$$=\overline{i}(0-0)-\overline{j}(-2-0)+\overline{k}(6x-4y)$$

$$=2\overline{j}+(6x-4y)\overline{k}$$

Since the projection is in xy-plane, $\bar{n} = \bar{k}, ds = dxdy$

$$\int_S curl\bar{F} \cdot \bar{n} ds = \int_S 2\bar{j} + (6x - 4y)\bar{k} \cdot \bar{k} dxdy = \iint_R (6x - 4y) dxdy$$



$$= \int_{x=0}^{2} \int_{y=0}^{x} (6x - 4y) dydx$$

$$= \int_{x=0}^{2} [6xy - 2y^2]_0^x dx = \int_{x=0}^{2} (6x^2 - 2x^2) dx$$

$$= [4\frac{x^3}{3}]_0^2 = \frac{32}{3}$$

Practice Problem

1.Verify Stoke's theorem for $\overline{F} = x^2\overline{i} + xy\overline{j}$ round the square in the plane z=0 whose sides along the lines x=0,y=0 x=a,y=a